

Long wavelength limit of evolution of nonlinear cosmological perturbations

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Abstract

In the general matter composition where the multiple scalar fields and the multiple perfect fluids coexist, in the leading order of the gradient expansion, we construct all the solutions of the nonlinear evolutions of the locally homogeneous universe. From the momentum constraint, we derive the constraints which the solution constants of the locally homogeneous universe must satisfy. We construct the gauge invariant perturbation variables in the arbitrarily higher order nonlinear cosmological perturbation theory around the spatially flat FRW universe. We construct the nonlinear LWL formula representing the long wavelength limit of the evolution of the nonlinear gauge invariant perturbation variables in terms of perturbations of the evolutions of the locally homogeneous universe. By using the LWL formula, we investigate the evolution of nonlinear cosmological perturbations in the universe dominated by the multiple slow rolling scalar fields with an arbitrary potential. The τ function and the N potential introduced in this paper make it possible to write the evolution of the multiple slow rolling scalar fields with an arbitrary interaction potential and the arbitrarily higher order nonlinear Bardeen parameter at the end of the slow rolling phase analytically. It is shown that the nonlinear parameters such as f_{NL} , g_{NL} are suppressed by the slow rolling expansion parameters.

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§1 Introduction and summary

In the inflationary universe scenario, the quantum fluctuations of the scalar fields called inflatons are thought to be the origin of the cosmological large scale structures such as galaxies and clusters of galaxies. In the slow rolling phase, the quantum fluctuations are stretched into the superhorizon scales and stay outside the horizon until they return into the horizon in the radiation dominant universe. Therefore the method for investigating the evolutions of the long wavelength cosmological perturbations became necessary and the LWL method was developed [27] [13] [24]. In the LWL method, we use the LWL formulae representing the long wavelength limit of the evolutions of the cosmological perturbations in terms of the quantities of the corresponding exactly homogeneous universe. As for the adiabatic modes the exact LWL formula had already been constructed and it was used in order to investigate the slow rolling phase [22] and the oscillatory phase [12] [6]. Nambu and Taruya suggested in the multiple scalar fields system also, the LWL formula exists [27]. Soon in this case the exact LWL formula was constructed [13] [24] and it was used to investigate the multiple oscillatory scalar fields [7] [8]. In addition, in the paper [13], we presented the flexible way for constructing the LWL formulae in the general matter composition. It will be called the Kodama and Hamazaki (KH) construction in this paper. In the KH construction, the perturbation variables related with the exactly homogeneous universe, such as the scalar field perturbation and the energy density perturbation, are expressed in terms of the exactly homogeneous perturbations, that is the derivative of the exactly homogeneous quantity with respect to the solution constant. By solving the spatial components of the Einstein equations, the perturbation variables not related with the exactly homogeneous quantities such as vector quantities, for example velocity perturbation variables, are expressed in the form of the integral of the perturbation variables related with the exactly homogeneous universe which have already been determined. We pointed out that the perturbation solution constants contained in the expressions of the perturbation variables determined in the processes explained above must satisfy the constraint coming from the momentum constraint. The KH construction can be applied to the system containing the perfect fluid components having vector degrees of freedom such as the velocity perturbation variables. Therefore by using the KH construction, in the most general matter composition where multiple scalar fields and multiple perfect fluids coexist, the LWL formula was constructed and used to investigate the multiple component reheating and the multiple component curvaton decay [9]. Afterward in case of the nonlinear perturbation also, the existence of the LWL formula was suggested [15] [23]. In this paper we give the definition of an arbitrarily higher order nonlinear gauge invariant perturbation variables and the exact nonlinear LWL formula representing them in terms of the derivatives of the quantities of the locally homogeneous universe with respect to the solution constants.

In order to calculate the present density perturbations and the cosmic microwave background anisotropies from the initial seed perturbations, we need to calculate the evolution of the scalar fields perturbations on superhorizon scales during the slow rolling phase. In the papers [5] [4], by decomposing the multiple scalar fields into the adiabatic field and the entropy fields instant by instant, the evolutionary behaviors of cosmological perturbations during the slow rolling phase were discussed. But this study is the local investigation, that is, it is based on the Taylor expansion of the evolution equations around the first horizon

crossing and the evolutions of the scalar fields in the long time interval were not solved. Then in this paper we present the method which makes it possible to trace the evolutions of the multiple slow rolling scalar fields with an arbitrary interaction potential for the long time period as in the whole slow rolling phase.

This paper is organized as follows. The section 2 is devoted to the nonlinear LWL formula. In the subsection 2.1, under the assumption that the universe is the spatial flat FRW universe in the background level, in the leading order of the gradient expansion, we give the evolution equations of the locally homogeneous universe. We point out that these locally homogeneous evolution equations are similar to the corresponding exactly homogeneous evolution equations and that deviations between the two are induced by the unimodular factor of the spatial metric $\tilde{\gamma}_{ij}$ which has contribution from the adiabatic decaying mode. Following the philosophy of the KH construction [13], we construct all the solutions of the evolution equations of the locally homogeneous universe. We give the constraint which comes from the momentum constraint and which must be satisfied by the solution constants contained in the solution of the locally homogeneous universe. In the subsection 2.2, we give the definition of the gauge invariant perturbation variables of the arbitrarily higher order nonlinear cosmological perturbation theory, and prove their gauge invariance. We present the nonlinear LWL formula representing the long wavelength limit of the evolutions of the gauge invariant perturbation variables in the nonlinear perturbation theory in terms of the derivatives with respect to the solution constant of the locally homogeneous solutions. The section 3 is devoted to the analysis of the nonlinear evolution of the multiple slow rolling scalar fields as an application of the nonlinear LWL formula constructed in the previous section. In the subsection 3.1, it is shown that in the slow rolling phase the scalar fields evolution equations are simplified by truncation. By estimating the truncation error, we establish the accuracy of the truncated evolution equations in the slow rolling expansion scheme. In the subsection 3.2, the ideas of the τ function and the N potential are introduced and they are shown to enable us to trace the multiple slow rolling scalar fields in the whole slow rolling phase analytically. By adopting the τ function as the evolution parameter, the truncated evolution equations of the multiple slow rolling scalar fields are simplified enough for their solutions to be written analytically. From the scalar fields solutions we can easily calculate the N potential which allows us to calculate the arbitrarily higher order Bardeen parameter at the end of the slow rolling phase from the initial scalar fields perturbations at the first horizon crossing. In the subsection 3.3, by the τ function and the N potential we calculate the various perturbation variables such as the Bardeen parameter, the entropy perturbations, the gravitational wave perturbation and their spectrum indices in the case of the multiple quadratic potential whose truncated evolution can be exactly solved. In the subsection 3.4, we consider the effect of the interaction between multiple slow rolling scalar fields in the case where the masses of the scalar fields do not satisfy any resonant relations. We point out that this problem is related with the well known Poincaré theorem about the linearization. In the concrete model, we calculate the various quantities such as the N potential and the nonlinear parameters f_{NL} , g_{NL} introduced in the paper [14]. In the subsection 3.5, we investigate the resonant interaction between the multiple slow rolling scalar fields. We show that the N potential has no singular part by introducing the resonant interactions. The section 4 is devoted to the discussions. The appendices are devoted to the proofs of

the propositions presented in the main content of this paper.

§2 Nonlinear LWL formula

In this section, we derive the nonlinear LWL formula in the most general matter composition whose energy momentum tensor is divided into $A = (S, f)$ parts where S represents N_S components scalar fields ϕ_a ($a = 1, 2, \dots, N_S$) and f represents N_f components perfect fluids ρ_α ($\alpha = 1, 2, \dots, N_f$). The content of this section is the nonlinear generalization of the content of the paper [9]. Our results in this section can be applied not only to the multiple slow rolling scalar fields, but also to the multicomponents reheating and the multicomponents curvaton decay [20] [9].

In the subsection 2.1, we give the evolution equations of the locally homogeneous universe in the leading order of the gradient expansion and we construct all the solutions of these locally homogeneous universe evolution equations. In the subsection 2.2, we give the definition of the nonlinear gauge invariant perturbation variables and we give the nonlinear LWL formula representing the long wavelength limit of the evolutions of the gauge invariant perturbation variables in terms of the derivative with respect to solution constant of the evolutions of the locally homogeneous universe determined in the subsection 2.1.

Our theory has two small expansion parameters: One is ϵ characterizing the small spatial derivative, that is, the small wavenumber, and the other is δ_c characterizing the amplitudes of the higher order perturbations. In general, under our present scheme, all the terms are classified as $O(\epsilon^k \delta_c^l)$ where k, l are appropriate nonnegative integers. Since we are interested in the full nonlinear perturbations, in the subsection 2.1 we will not Taylor expand with respect to δ_c . But since we treat the evolutions of the long wavelength perturbations only, we will expand with respect to ϵ and we will drop terms which are small compared to the leading order by $O(\epsilon^2)$ order quantity. This process corresponds to the leading order of the gradient expansion [23] [15] [26] and the universe treated in this way is called the locally homogeneous universe. The locally homogeneous evolution equations are very similar to the exactly homogeneous evolution equations. The evolutions of locally homogeneous physical quantities which have counterparts in the corresponding exactly homogeneous universe can be determined as easily as the exactly homogeneous evolutions are determined. This is the attractive point of our LWL method.

In our scheme, we can assign $\partial_t \tilde{\gamma}_{ij} = O(\delta_c)$, $\partial_t^2 \tilde{\gamma}_{ij} = O(\delta_c)$ where $\tilde{\gamma}_{ij}$ is the unimodular factor of the spatial metric defined by (2.5), since we require only that the universe should be the spatially flat FRW universe in the background level. So in the leading order of the gradient (ϵ) expansion the terms containing $\partial_t \tilde{\gamma}_{ij}$, $\partial_t^2 \tilde{\gamma}_{ij}$ cannot be dropped. But in the papers [15] [26], mainly in order to avoid the computational complexity, the authors assigned $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$, $\partial_t^2 \tilde{\gamma}_{ij} = O(\epsilon^2)$ and discarded the terms containing $\partial_t \tilde{\gamma}_{ij}$, $\partial_t^2 \tilde{\gamma}_{ij}$. But this is confusion between the different small parameters ϵ , δ_c and cannot be justified. In this section, under the more natural assumption $\tilde{\gamma}_{ij} = \delta_{ij} + O(\delta_c)$, we show that the evolution of $\tilde{\gamma}_{ij}$ can be solved analytically and that the evolution equations of the other dynamical variables are simple enough to be solved analytically in spite of inclusion of the terms containing $\partial_t \tilde{\gamma}_{ij}$, $\partial_t^2 \tilde{\gamma}_{ij}$. By doing so, in the leading order of the gradient (ϵ) expansion, the evolutions of all the modes $2N_S + 4N_f + 4$ are obtained consistently in the universe where N_S scalar fields and N_f perfect fluids coexists. In the papers [15] [26],

the evolutions of the several dynamical variables become higher order $O(\epsilon^2)$, therefore the initial conditions of such variables are unnaturally constrained to too small quantities.

2.1 Evolution of the locally homogeneous universe

First we present the Einstein equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$ where $\kappa^2 = 8\pi G$ is the gravitational constant, based on the 3 + 1 decomposition based on the paper [25]. The metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

is written by

$$g_{00} = -\alpha^2 + \beta_k \beta^k, \quad (2.2)$$

$$g_{0i} = \beta_i, \quad (2.3)$$

$$g_{ij} = \gamma_{ij}, \quad (2.4)$$

where α is the lapse and β_i is the shift vector and $\beta^i := \gamma^{ij} \beta_j$. Greek indices take the values $\mu, \nu = 0, 1, 2, 3$ and Latin indices take values $i, j = 1, 2, 3$. The spatial metric γ_{ij} is decomposed as

$$\gamma_{ij} = a^2 \tilde{\gamma}_{ij}, \quad \det(\tilde{\gamma}_{ij}) = 1, \quad (2.5)$$

where a can be interpreted as the nonlinear generalization of the scale factor. The extrinsic curvature of the $t = \text{const}$ hypersurface is given by

$$\begin{aligned} -K_{ij} &= \alpha \Gamma_{ij}^0 \\ &= \frac{1}{2\alpha} (\dot{\gamma}_{ij} - D_i \beta_j - D_j \beta_i), \end{aligned} \quad (2.6)$$

where D_i is the covariant derivative with respect to γ_{ij} . The extrinsic curvature is decomposed as

$$K_{ij} = \frac{1}{3} \gamma_{ij} K + a^2 \tilde{A}_{ij}, \quad (2.7)$$

$$\gamma^{ij} \tilde{A}_{ij} = 0. \quad (2.8)$$

Then we obtain

$$-K = \frac{1}{\alpha} \left(3 \frac{\dot{a}}{a} - D_i \beta^i \right). \quad (2.9)$$

The energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu}, \quad (2.10)$$

where ρ , P , u_μ are the energy density, the pressure, and the 4 velocity of the total system, respectively. The 4 velocity u^μ is written as

$$u^0 = [\alpha^2 - (\beta_k + v_k)(\beta^k + v^k)]^{-1/2}, \quad (2.11)$$

$$u^i = u^0 v^i, \quad (2.12)$$

where v_i is the 3 velocity of the total system and $v^i := \gamma^{ij}v_j$. By using $n_\mu := (-\alpha, 0, 0, 0)$ which is the unit vector normal to the time slices, the 3 + 1 decomposition of the energy momentum tensor is given by

$$E := T_{\mu\nu}n^\mu n^\nu = (\rho + P)(\alpha u^0)^2 - P, \quad (2.13)$$

$$J_j := -T_{\mu\nu}n^\mu \gamma_j^\nu = (\rho + P)\alpha u^0 u_j, \quad (2.14)$$

$$S_{ij} := T_{ij} = (\rho + P)(u^0)^2(\beta_i + v_i)(\beta_j + v_j) + P\gamma_{ij}. \quad (2.15)$$

The Hamiltonian and momentum constraints are written as

$$R - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 2\kappa^2 E, \quad (2.16)$$

$$D_i\tilde{A}^i_j - \frac{2}{3}D_j K = \kappa^2 J_j, \quad (2.17)$$

where the indices of \tilde{A}_{ij} is raised by $\tilde{\gamma}^{ij}$ which is the inverse matrix of $\tilde{\gamma}_{ij}$. The evolution equations for γ_{ij} are written as

$$(\partial_t - \beta^k \partial_k) a = \frac{1}{3}a(-\alpha K + \partial_k \beta^k), \quad (2.18)$$

$$(\partial_t - \beta^k \partial_k) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j \beta^k + \tilde{\gamma}_{jk}\partial_i \beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\partial_k \beta^k. \quad (2.19)$$

The evolution equations for K_{ij} are given by

$$\begin{aligned} (\partial_t - \beta^k \partial_k) K &= \alpha \left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2 \right) \\ &\quad - D_k D^k \alpha + \frac{\kappa^2}{2}\alpha(E + S_k^k), \end{aligned} \quad (2.20)$$

$$\begin{aligned} (\partial_t - \beta^k \partial_k) \tilde{A}_{ij} &= \frac{1}{a^2} \left[\alpha \left(R_{ij} - \frac{1}{3}\gamma_{ij}R \right) - \left(D_i D_j \alpha - \frac{1}{3}\gamma_{ij}D_k D^k \alpha \right) \right] \\ &\quad + \alpha \left(K \tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}_j^k \right) + \tilde{A}_{ik}\partial_j \beta^k + \tilde{A}_{jk}\partial_i \beta^k \\ &\quad - \frac{2}{3}\tilde{A}_{ij}\partial_k \beta^k - \frac{\kappa^2 \alpha}{a^2} \left(S_{ij} - \frac{1}{3}\gamma_{ij}S_k^k \right). \end{aligned} \quad (2.21)$$

R_{ij} is the Ricci tensor of the metric γ_{ij} and $R = \gamma^{ij}R_{ij}$, $S_k^k = \gamma^{kl}S_{kl}$.

Next under the 3 + 1 decomposition, we rewrite the evolution equations of the energy momentum tensor $T_{A\nu}^\mu$ of the A component:

$$\nabla_\nu T_{A\mu}^\nu = Q_{A\mu}, \quad (2.22)$$

where $A := (a, \alpha)$, a is the scalar fields index and α is the perfect fluids index. The energy momentum transfer vector $Q_{A\mu}$ is decomposed as

$$Q_{A\mu} = Q_A u_\mu + f_{A\mu}, \quad (2.23)$$

$$u^\mu f_{A\mu} = 0, \quad (2.24)$$

where Q_A , $f_{A\mu}$ are the energy transfer, the momentum transfer vector of the A component, respectively. The energy momentum tensor of the total system T^μ_ν can be expressed as the sum of the each component energy momentum tensor $T^\mu_{A\nu}$:

$$T^\mu_\nu = \sum_A T^\mu_{A\nu} \quad (2.25)$$

The energy momentum conservation, that is, the Bianchi identity gives

$$\sum_A Q_{A\mu} = 0. \quad (2.26)$$

We consider the scalar field component $\phi = (\phi_a)$. The energy momentum tensor of the scalar fields part is given by

$$(T^\mu_\nu)_S = \nabla_\mu \phi \cdot \nabla_\nu \phi - \frac{1}{2} [g^{\rho\sigma} \nabla_\rho \phi \cdot \nabla_\sigma \phi + 2U] g_{\mu\nu} \quad (2.27)$$

The energy momentum tensor of the scalar fields part $(T_{\mu\nu})_S$ can be written as the perfect fluid form by identifying

$$\rho_S = -\frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \cdot \nabla_\sigma \phi + U, \quad (2.28)$$

$$P_S = -\frac{1}{2} g^{\rho\sigma} \nabla_\rho \phi \cdot \nabla_\sigma \phi - U, \quad (2.29)$$

$$u_{a\mu} = -\frac{1}{\text{sign}(\dot{\phi}_a) (-g^{\rho\sigma} \nabla_\rho \phi \cdot \nabla_\sigma \phi)^{1/2}} \partial_\mu \phi_a, \quad (2.30)$$

where the minus sign of $u_{a\mu}$ is adopted by requiring $u_{a0} = -\alpha + O(\epsilon^2)$ in the gradient expansion scheme, where ϵ is the small parameter characterizing the spatial derivative defined below. When we assume that the energy momentum transfer vector of the scalar fields part $(Q_\mu)_S$ is given by

$$(Q_\mu)_S = S_a \nabla_\mu \phi_a, \quad (2.31)$$

where the source function S_a describes the energy transfer from the scalar field ϕ_a to other components, the evolution equation of the scalar fields components $\nabla_\nu (T^\nu_\mu)_S = (Q_\mu)_S$ holds if the scalar field ϕ_a satisfies the phenomenological equations of motion of the scalar field ϕ_a :

$$\square \phi_a - \frac{\partial U}{\partial \phi_a} = S_a, \quad (2.32)$$

where

$$\square \phi_a = (-g)^{-1/2} \partial_\mu \left[(-g)^{1/2} g^{\mu\nu} \partial_\nu \phi_a \right], \quad (2.33)$$

$$g := \det(g_{\mu\nu}). \quad (2.34)$$

Since we are interested in the cosmological perturbations on superhorizon scales, we put the gradient expansion assumption defined by

$$\partial_i = O(\epsilon), \quad \beta_i = O(\epsilon), \quad v_i = O(\epsilon), \quad f_{Ai} = O(\epsilon), \quad (2.35)$$

where ϵ is the small parameter characterizing the small wavenumber of the cosmological perturbations. By the assumption $\partial_i = O(\epsilon)$, we assume that the spatial scale of all the inhomogeneities is of the order of $1/\epsilon$, that is, all the physical quantities which are approximately homogeneous on each horizon can vary on the superhorizon scales. The local homogeneity and isotropy in the horizon guarantee that the vector quantities such as β_i , v_i , f_{Ai} are of the order of $O(\epsilon)$. Unlike the papers [15] [26] where $\partial_t \tilde{\gamma}_{ij} = O(\epsilon^2)$ is assumed, from the requirement that our locally homogeneous universe should be the spatially flat FRW universe in the background level, we assume that

$$\tilde{\gamma}_{ij} = \delta_{ij} + O(\delta_c). \quad (2.36)$$

where δ_c is the small parameter characterizing the higher order perturbations. Since we consider the leading order of the gradient (ϵ) expansion without expanding with respect to δ_c , we must keep the terms containing $\partial_t \tilde{\gamma}_{ij} = O(\delta_c)$, $\partial_t^2 \tilde{\gamma}_{ij} = O(\delta_c)$ as explained in the beginning of this section.

Under the gradient (ϵ) expansion scheme, the relations between the total system quantities and the each component quantities are given by

$$\rho = \sum_A \rho_A, \quad (2.37)$$

$$P = \sum_A P_A, \quad (2.38)$$

$$h = \sum_A h_A, \quad (2.39)$$

$$h v_i = \sum_A h_A v_{Ai} + O(\epsilon^3), \quad (2.40)$$

$$0 = \sum_A Q_A, \quad (2.41)$$

$$0 = \sum_A f_{Ai}, \quad (2.42)$$

where h_A is the A component enthalpy defined by $h_A := \rho_A + P_A$.

As for the fluid component α , $\nabla_\mu T_{\alpha 0}^\mu = Q_{\alpha 0}$ gives

$$\dot{\rho}_\alpha = -3H(\rho_\alpha + P_\alpha) + Q_\alpha \alpha + O(\epsilon^2), \quad (2.43)$$

where H is the Hubble parameter defined by $H := \dot{a}/a$, and integrating $\nabla_\mu T_{\alpha i}^\mu = Q_{\alpha i}$ with respect to t gives

$$h_\alpha(\beta_i + v_{\alpha i}) = \frac{\alpha}{a^3} C_{\alpha i} + \frac{\alpha}{a^3} \int_{t_0} dt \alpha a^3 \left[-\partial_i P_\alpha - \frac{1}{\alpha} D_i \alpha h_\alpha + Q_{\alpha i} \right] + O(\epsilon^3), \quad (2.44)$$

where t_0 is the initial time and $C_{\alpha i} := C_{\alpha i}(\mathbf{x})$ are the integration constants.

As for the scalar field components, from (2.32), the equation of motion of the scalar field ϕ_a is given by

$$\frac{1}{\alpha^2} \left[\ddot{\phi}_a + 3 \frac{\dot{a}}{a} \dot{\phi}_a - \frac{\dot{\alpha}}{\alpha} \dot{\phi}_a \right] + \frac{\partial U}{\partial \phi_a} + S_a = O(\epsilon^2). \quad (2.45)$$

Since the source function S_a is the scalar quantity, we can assume that S_a is the function of other scalar quantities. As such scalar quantities, we can adopt ϕ_a ,

$$T_{2a} := \text{sgn}(\partial_0 \phi_a) (-g^{\rho\sigma} \nabla_\rho \phi_a \nabla_\sigma \phi_a)^{1/2} = \frac{1}{\alpha} \dot{\phi}_a + O(\epsilon^2), \quad (2.46)$$

or

$$T_{2a} := u^\mu \nabla_\mu \phi_a = \frac{1}{\alpha} \dot{\phi}_a + O(\epsilon^2) \quad (2.47)$$

where u_μ is the arbitrary unit time like vector field and

$$T_3 := \nabla_\mu u^\mu = 3 \frac{1}{\alpha} \frac{\dot{a}}{a} + O(\epsilon^2). \quad (2.48)$$

Therefore in the leading order of the ϵ expansion, the form of the source functions S_a can be given by

$$S_a = S_a \left(\phi_a, \frac{1}{\alpha} \dot{\phi}_a, 3 \frac{1}{\alpha} \frac{\dot{a}}{a} \right) + O(\epsilon^2). \quad (2.49)$$

For example,

$$S_a = \Gamma_a \frac{1}{\alpha} \dot{\phi}_a, \quad (2.50)$$

where Γ_a is the decay constant of the scalar field ϕ_a , is the most simple source function.

Since until now we have presented the leading order of the gradient (ϵ) expansion of all the locally homogeneous evolution equations, we will construct all the solutions of the evolutions of the locally homogeneous universe. From (2.19), we obtain

$$\tilde{A}_{ij} = -\frac{1}{2\alpha} \frac{d}{dt} \tilde{\gamma}_{ij} + O(\epsilon^2). \quad (2.51)$$

By substituting the above equation into (2.21), we obtain

$$\frac{d^2}{dt^2} M = \left(\frac{d}{dt} \ln \frac{\alpha}{a^3} \right) \dot{M} + \dot{M} M^{-1} \dot{M} + O(\epsilon^2), \quad (2.52)$$

where $M := (\tilde{\gamma}_{ij})$. By neglecting $O(\epsilon^2)$ order terms, we obtain the solution as

$$M = R \exp \left[\int_{t_0} dt \frac{\alpha}{a^3} T \right], \quad (2.53)$$

where $R = R(\mathbf{x})$, $T = T(\mathbf{x})$ are time independent matrices. Since M is a unimodular symmetric matrix for an arbitrary t , R is unimodular symmetric, T is traceless, and RT is symmetric. By using (2.53), we obtain

$$\tilde{A}_{ij} \tilde{A}^{ij} = \frac{1}{4\alpha^2} \text{tr} \left(\dot{M} M^{-1} \dot{M} M^{-1} \right) = \frac{c_T}{a^6}, \quad (2.54)$$

$$c_T := \frac{1}{4} \text{tr} (T^2), \quad (2.55)$$

where $c_T = c_T(\mathbf{x})$ is a time independent constant. By using the above results, (2.16) gives

$$H^2 = \alpha^2 \left(\frac{\kappa^2}{3} \rho + \frac{1}{6} \frac{c_T}{a^6} \right), \quad (2.56)$$

(2.20) gives

$$\dot{H} = \frac{\dot{\alpha}}{\alpha} H - \frac{\alpha^2 c_T}{2 a^6} - \frac{\alpha^2 \kappa^2}{2} (\rho + P). \quad (2.57)$$

By eliminating H in (2.56) and (2.57), we obtain the well known continuity equation of the total system in the expanding universe as

$$\dot{\rho} = -3H(\rho + P). \quad (2.58)$$

Eqs.(2.43)(2.45) and the above three evolution equations agree with the exactly homogeneous evolution equations if the proper time slicing $\alpha = 1$ [10] and $c_T = 0$. But we cannot assume $c_T = 0$ since the solution constants must satisfy the constraint originating from the momentum constraint (2.17) as

$$\kappa^2 \sum_{\alpha} C_{j\alpha} + \frac{1}{2} \partial_i T_j^i - \frac{1}{4} \text{tr} [R^{-1} \partial_j R T] - \left[2a^3 \partial_j \left(\frac{H}{\alpha} \right) + \kappa^2 \frac{a^3}{\alpha} \sum_a \dot{\phi}_a \partial_j \phi_a \right]_{t_0} = 0. \quad (2.59)$$

For the derivation of (2.59), see Appendix A.

We consider the long wavelength limit of all the solutions of the evolution equations of the locally homogeneous universe. The unimodular factor of the spatial metric $\tilde{\gamma}_{ij}$ is obtained by (2.53). These $\tilde{\gamma}_{ij}$ induce the deviation between the true locally homogeneous universe and the corresponding exactly homogeneous universe, for example c_T terms in (2.56) and (2.57). They contain the scalar adiabatic decaying mode which was carefully treated in the papers [13] [9], since in the scalar part in the linear cosmological perturbation theory it induces the deviation between the long wavelength limit of the true perturbation solutions and the derivative of the exactly homogeneous universe with respect to the solution constant. According to the philosophy of the KH construction [13], the expressions of ρ_{α} , ϕ_a related with the exactly homogeneous quantities are obtained by solving the evolution equations (2.43) (2.45) under the Hamiltonian constraint (2.56) and the proper time slicing $\alpha = 1$. Under the proper time slicing $\alpha = 1$, (2.43), (2.45) and (2.56) are almost the same as the counterparts of the exactly homogeneous universe. Therefore solving the former true locally homogeneous evolution equations requires as little labor as solving the latter exactly homogeneous evolution equations. As for the velocity perturbations of the fluid components $h_{\alpha}(\beta_i + v_{\alpha i})$ which are the vector quantities not related with the exactly homogeneous quantities, their evolutions are given by (2.44) in whose right hand side the second integral term contains P_{α} , h_{α} , α which have already been determined by the previous process. The solution constants must satisfy the momentum constraint (2.59).

Let us count the degrees of freedom. As for (2.53), R is unimodular symmetric, T is traceless, and RT is symmetric, therefore R , T have 5 degrees of freedom, respectively. But by the coordinate transformation $\bar{x}^i = f^i(x)$ ($i = 1, 2, 3$) the 3 degrees of freedom of R can be made vanishing. According to (2.43) (2.45), the densities ρ_{α} and the scalar fields ϕ_a have N_f , $2N_S$ degrees of freedom, respectively. According to (2.44), the fluids velocities $v_{\alpha i}$ have $3N_f$ degrees of freedom. The momentum constraint (2.59) gives 3 constraints. Therefore the total degrees of freedom is $5 + 5 - 3 + N_f + 2N_S + 3N_f - 3 = 4 + 4N_f + 2N_S$. Then we have obtained all the solutions of the evolution equations of the locally homogeneous universe in the leading order of the gradient (ϵ) expansion.

From the time dependence of all the solutions, we can interpret the physical roles of all the solutions. 2 from R can be interpreted as the gravitational wave growing modes. 5 from T can be interpreted as the 2 gravitational wave decaying modes, the 1 adiabatic scalar decaying mode and the 2 adiabatic vector decaying modes. By the 3 momentum constraints (2.59), 3 of $C_{j\alpha}$ are adjusted. In the remaining $(3N_f - 3)$ $C_{j\alpha}$'s, the $(2N_f - 2)$ entropic vector decaying modes and the $(N_f - 1)$ entropic scalar decaying modes are contained. The N_f densities ρ_α have the N_f scalar growing modes and the N_S scalar fields ϕ_a have the N_S scalar fields growing modes and the N_S scalar fields decaying modes.

2.2 Gauge invariant variables and the derivation of the LWL formula

In this subsection, we give the definitions of the gauge invariant perturbation variables in the arbitrary higher order perturbation theory in the leading order of the gradient (ϵ) expansion and the LWL formula representing the evolutions of these gauge invariant perturbation variables in terms of derivative with respect to the solution constant of the corresponding physical quantities of the locally homogeneous universe.

As for $M = (\tilde{\gamma}_{ij})$, since the homogeneous parts of R , T defined by (2.53) are determined by the fact that the background metric is the spatially flat FRW universe, R , T are expanded as

$$R_{ij}(\mathbf{x}) = \delta_{ij} + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k R_{ij}(\mathbf{x}), \quad (2.60)$$

$$T_{ij}(\mathbf{x}) = \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k T_{ij}(\mathbf{x}), \quad (2.61)$$

where δ in the above is the operator generating the higher order perturbation quantities.

As shown in the previous subsection, in the leading order of the gradient (ϵ) expansion, the evolution equations of the locally homogeneous universe which have counterparts in the evolution equations of the exactly homogeneous universe do not contain the spatial derivative. Therefore in the expression of the solution of the evolution of an arbitrary locally homogeneous physical quantity (related with the exactly homogeneous quantity) A such as the scalar fields ϕ_a and the fluid energy densities ρ_α , all the dependences on the spatial coordinate \mathbf{x} are contained in the (time independent) spatial dependent integration constants $C(\mathbf{x})$:

$$A = A(t, C(\mathbf{x})). \quad (2.62)$$

Since the solutions of the evolution equations of the locally homogeneous universe contain the nonlinear effect in the full order, the locally homogeneous physical quantity A can be expanded as

$$A(t, \mathbf{x}) = A(t) + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k A(t, \mathbf{x}). \quad (2.63)$$

The higher order perturbation effects are induced by the dependences on the spatial coor-

dinate \mathbf{x} . Therefore the each solution constant $C_i(\mathbf{x})$ can be expanded as

$$C_i(\mathbf{x}) = C_i + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k C_i(\mathbf{x}), \quad (2.64)$$

whose background part is spatially independent. The k -th order nonlinear perturbation $\delta^k A$ can be expressed in terms of the nonlinear perturbations of the spatial dependent integration constants $\delta^l C(\mathbf{x})$. In this paper, the expressions representing $\delta^k A$ in terms of $\delta^l C(\mathbf{x})$ are called the LWL formula. In order to derive the LWL formula, we propose a simple mathematical trick. In the leading order of the gradient (ϵ) expansion, the expression of the solution of an arbitrary locally homogeneous physical quantity A can be written as function of the time t and the integration constants C as shown in (2.62). We assume that all the integration constants C depend on one parameter λ imaginarily instead of \mathbf{x} . As understood below, λ represents the spatial coordinate dependence symbolically. The physical quantity A can be expanded as

$$A(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \frac{d^k}{d\lambda^k} A(\lambda) \Big|_{\lambda=0}, \quad (2.65)$$

The full order nonlinear solution is formally recovered by setting $\lambda = 1$. Since both the perturbation δ and the λ differentiation $d/d\lambda$ are the derivative operators satisfying the same chain and product rules, and the k -th order λ differentiation $d^k/d\lambda^k \dots|_{\lambda=0}$ is multiplied by λ^k , we can use the λ differentiation to track the algebraic behaviors of the perturbation δ . By comparing (2.65) with (2.63), we can read the correspondences given by

$$\frac{d^k}{d\lambda^k} A(\lambda) \Big|_{\lambda=0} \leftrightarrow \delta^k A(t, \mathbf{x}), \quad (2.66)$$

The gauge transformation can be expressed by the Lie derivative $L(T)$ as

$$A(\lambda, \mu) = \exp\{\mu L(T)\} A(\lambda, \mu = 0), \quad (2.67)$$

where μ is the parameter characterizing the size of the gauge transformation and $A(\lambda, \mu = 1)$ is the transformed variable and $A(\lambda, \mu = 0) = A(\lambda)$ is the original variable. When the vector field $T := T^\mu \partial_\mu$ in the Lie derivative $L(T)$ can be expanded in terms of λ , the zeroth order term $T(\lambda = 0)$ is zero since we consider the infinitesimal gauge transformation. By differentiating (2.67) with respect to λ , afterward putting $\lambda = 0$, we obtain the well known expressions of the gauge transformations of $\delta^k A$ [3]:

$$\tilde{\delta} A = L_1 A + \delta A, \quad (2.68)$$

$$\delta^2 A = L_2 A + L_1 L_1 A + 2L_1 \delta A + \delta^2 A, \quad (2.69)$$

where $\tilde{\delta}^n A$ is the gauge transformed perturbation variables of $\delta^n A$, and L_k is the Lie derivative induced by $\delta^k T$; $L_k := L(\delta^k T)$. The gauge transformation law (2.67) is the solution of the differential equation:

$$\frac{d}{d\mu} A(\lambda, \mu) = L(T) A(\lambda, \mu), \quad (2.70)$$

which is much simpler than the individual gauge transformation expressions of $\delta^k A$. As for the vector field $T := T^\mu \partial_\mu$ inducing the Lie derivative $L(T)$, we can assume that

$$T^i = O(\epsilon), \quad (2.71)$$

since we consider only the gauge transformations keeping the local homogeneity and isotropy in the horizon.

By using the differential equation (2.70), we can show the following proposition:

Proposition 1 *For an arbitrary scalar quantity A , the perturbation quantities $D^n A$, $D^n(\dot{A}/\alpha)$ where D is defined by*

$$D := \frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt}, \quad (2.72)$$

are gauge invariant up to order $O(\epsilon^2)$ error. For the lapse function α , \mathcal{A}_n defined by

$$\mathcal{A}_n := \left(\frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt} \right)^n \alpha. \quad (2.73)$$

is gauge invariant up to order $O(\epsilon^2)$ error.

For the proof, see Appendix C.

$D^n A$, $D^n(\dot{A}/\alpha)$ and \mathcal{A}_n are higher order generalization of DA , $D\dot{A}$ and \mathcal{A} defined in the paper [9], respectively. $D^n(\dot{A}/\alpha)$ and \mathcal{A}_n are not independent. For example

$$D \left(\frac{\dot{A}}{\alpha} \right) = \frac{1}{\alpha^2} \left\{ (DA) \dot{\alpha} - \dot{A} \mathcal{A}_1 \right\}. \quad (2.74)$$

Therefore by putting $A = \phi_a$, $A = \rho_\alpha$, we can use $D^n \phi_a$, $D^n(\dot{\phi}_a/\alpha)$, $D^n \rho_\alpha$ as the independent perturbation variables. Our DA is almost the same as the well known gauge invariant perturbation variable, for example, for $A = \phi_a$ the Sasaki Mukhanov variable in the linear perturbation theory [10] [11] [18] [9]. Our $D^2 A$ agrees almost perfectly with the gauge invariant quantities introduced by the second order gauge invariant perturbation theory [16] [17] [19]. The reason of the tiny deviation between our gauge invariant perturbation quantity and the gauge invariant perturbation quantity of the first and the second order gauge invariant perturbation theory is that we assume $T^i = O(\epsilon)$ based on the gradient expansion scheme. In our scheme (2.71), we can regard the scale factor a as the scalar quantity up to $O(\epsilon^2)$. Therefore in the same way as in the proof of Proposition 1 we can show that

$$\zeta_n := \bar{\delta}^n \ln a, \quad (2.75)$$

$$\bar{\delta} := \frac{d}{d\lambda} - \frac{d\rho}{d\lambda} \frac{1}{\dot{\rho}} \frac{d}{dt} \quad (2.76)$$

are gauge invariant up to $O(\epsilon^2)$. ζ_n is the higher order generalization of the well known Bardeen parameter [2] [28].

We discuss the influence of the replacement of the evolution parameter. Concretely we consider replacing the old evolution parameter t with the new evolution parameter as

the scale factor a . With the scale factor a as the evolution parameter, the D operation defined by (2.72) can be written in the more simple form. $d/d\lambda$ in (2.72) is the λ derivative with t fixed, that is, operating on the locally homogeneous physical quantity such as ρ_α , ϕ_a expressed by using t as the evolution parameter. However an arbitrary locally homogeneous quantities can be also expressed by using the scale factor a as the evolution parameter instead of t . In this case, we can consider the differentiation $(\partial/\partial\lambda)_a$ taken at fixed a . The differential operators in the two group $(d/d\lambda, d/dt)$ $((\partial/\partial\lambda)_a, (\partial/\partial a)_a)$ are commutative in the each group, but the differential operators belonging to the different groups, for example $d/d\lambda$ and $(\partial/\partial a)_a$, are not commutative. By using the relation

$$\frac{d}{d\lambda} = \left(\frac{\partial}{\partial\lambda} \right)_a + \frac{da}{d\lambda} \left(\frac{\partial}{\partial a} \right)_a, \quad (2.77)$$

the D operation (2.72) can be expressed much more simply as

$$D = \left(\frac{\partial}{\partial\lambda} \right)_a. \quad (2.78)$$

Therefore $D^n A$, $D^n(\dot{A}/\alpha)$ can be written in the very simple way as

$$D^n A = \left(\frac{\partial^n}{\partial\lambda^n} \right)_a A, \quad D^n \left(\frac{\dot{A}}{\alpha} \right) = \left(\frac{\partial^n}{\partial\lambda^n} \right)_a \left(\frac{\dot{A}}{\alpha} \right). \quad (2.79)$$

In this way, in the LWL formalism where the scale factor a , not t , is used as the evolution parameter [7] [8] [9], the expressions of the solutions of $D^n A$, $D^n(\dot{A}/\alpha)$ can be obtained by calculating only a single term written with the higher order $(\partial/\partial\lambda)_a$ derivative of the solution of the corresponding locally homogeneous physical quantity. In the same way, $\bar{\delta}$ defined by (2.76) can be expressed in the word of the scale factor a as

$$\bar{\delta} = \left(\frac{\partial}{\partial\lambda} \right)_a - \left\{ \left(\frac{\partial\rho}{\partial\lambda} \right)_a / \left(\frac{\partial\rho}{\partial a} \right)_a \right\} \left(\frac{\partial}{\partial a} \right)_a. \quad (2.80)$$

As for the Bardeen parameter defined by (2.75), we can show the following propositions.

Proposition 2 *If ζ_1 is conserved for arbitrary values of integration constants $C(\lambda = 0)$, all ζ_n ($n \geq 2$) are also conserved, and they can be expressed as*

$$\zeta_n = \left(\frac{\partial}{\partial\lambda} \right)_a^{n-1} \zeta_1 \quad (2.81)$$

For the proof, see Appendix D

Proposition 3 *When $P = P(\rho)$ holds, ζ_n ($n \geq 1$) can be expressed as*

$$\zeta_n = \frac{1}{3} \left(\frac{\partial^n}{\partial\lambda^n} \right)_a \left[\int d\rho \frac{1}{\rho + P(\rho)} \right] \quad (2.82)$$

and all ζ_n ($n \geq 1$) are conserved.

For the proof, see Appendix E

All the definitions of the nonlinear gauge invariant perturbation variables in this subsection are written in terms of the λ differentiations. As explained at the beginning of this subsection, the λ differentiation implies not only the symbol of the higher order nonlinear perturbation of the physical quantity, but also the process of taking the derivative with respect to the integration constants of the corresponding locally homogeneous quantity. Therefore all the definitions of the nonlinear gauge invariant perturbation variables in terms of the λ differentiations can also be regarded as the LWL formulae themselves, so from now on they will be called the LWL formulae.

§3 Nonlinear evolution of the multiple slow rolling scalar fields

In this section, as the application of the LWL formula derived in the previous section, we consider the evolutions of the long wavelength nonlinear cosmological perturbations in the universe dominated by the multiple slow rolling scalar fields. The τ function and the N potential introduced in this section are the useful tools for tracing analytically the evolutions of the multiple slow rolling scalar fields in the long time interval. We calculate spectral indices of the linear cosmological perturbations. In the interacting system, we derive the formulae giving the amplitudes of the nonlinear Bardeen parameters at the end of the slow rolling phase in terms of the initial scalar fields perturbations, and calculate the nonlinear parameters f_{NL} , g_{NL} [14] representing the non-Gaussianity of the Bardeen parameter.

3.1 Evolution of the multiple slow rolling scalar fields

In the slow rolling phase, the scalar fields ϕ_a roll slowly on the potential U . The potential energy U which hardly changes triggers the exponential expansion of the universe. The Hubble parameter is large compared to the masses of the scalar fields. The ratio of the kinetic energy part in the whole energy density ρ is small compared to the contribution from the potential energy U . In the investigations of the evolutions of the scalar fields under this situation, it is effective to use the transformation by which the evolution equations of the slow rolling scalar fields system are greatly simplified, that is the effects of the time derivatives of the scalar fields $p_a := \dot{\phi}_a/\alpha$ on the evolutions of the scalar fields ϕ_a are eliminated.

In this section, we consider the multiple slow rolling scalar fields ϕ_a under the conditions:

$$S_a = 0, \quad U = \sum_a \frac{1}{2} m_a^2 \phi_a^2 + U_{\text{int}}. \quad (3.1)$$

where U_{int} is the sum of m -th order monomials ($m \geq 3$). As the independent variables, we adopt ϕ_a and $p_a := \dot{\phi}_a/\alpha$. By nondimensionalizing the dynamical variables as

$$\frac{a}{a_0} \rightarrow a, \quad \frac{\phi_a}{\phi_0} \rightarrow \phi_a, \quad \frac{p_a}{p_0} \rightarrow p_a, \quad \frac{c_T}{c_{T0}} \rightarrow c_T, \quad (3.2)$$

and the parameter as

$$\frac{m_a}{m_0} \rightarrow m_a, \quad (3.3)$$

that is

$$\frac{\rho}{m_0^2 \phi_0^2} \rightarrow \rho, \quad (3.4)$$

we obtain the dimensionless parameters:

$$\epsilon_* := \frac{\sqrt{3}}{\kappa \phi_0}, \quad \eta^2 := \frac{p_0^2}{m_0^2 \phi_0^2}, \quad \nu^2 := \frac{1}{2} \frac{c_{T0}}{\kappa^2 m_0^2 \phi_0^2} \frac{1}{a_0^6}. \quad (3.5)$$

ϵ_* is the small constant representing the ratio of the mass scale to the Hubble parameter. This ϵ_* is different from ϵ characterizing the small wavenumber in the previous section. η^2 is positive constant since we consider only the scalar fields with positive definite kinetic parts. We assume that $\epsilon_* \sim \eta \sim \nu \ll 1$. Then the evolution equations of ϕ_a , p_a and c_T are given by

$$\frac{d}{dN} \phi_a = \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \eta^2 p_a, \quad (3.6)$$

$$\frac{d}{dN} p_a = -3p_a - \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \frac{\partial U}{\partial \phi_a}, \quad (3.7)$$

$$c_T = \text{const}, \quad (3.8)$$

where the evolution parameter is the e-folding number of the scale factor $N := \ln a$ and the energy density ρ is given by

$$\rho = \frac{\eta^2}{2} \sum_a p_a^2 + U + \nu^2 c_T \frac{1}{a^6}. \quad (3.9)$$

By using $p_a^{(1)}$ defined by

$$p_a^{(1)} := p_a + \frac{1}{3} \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \frac{\partial U}{\partial \phi_a}, \quad (3.10)$$

which represents the deviation of p_a from the truncated slow rolling solution, the evolution equations (3.6)(3.7) can be rewritten as

$$\frac{d}{dN} \phi_a = \eta^2 F_a(\phi) + \eta^2 f_a(\phi, p^{(1)}, c_T, N), \quad (3.11)$$

$$\frac{d}{dN} p_a^{(1)} = -3p_a^{(1)} + \eta^2 g_a(\phi, p^{(1)}, c_T, N) \quad (3.12)$$

where

$$F_a(\phi) := -\frac{1}{3} \frac{\epsilon_*^2}{\eta^2} \frac{1}{U} \frac{\partial U}{\partial \phi_a}, \quad (3.13)$$

and

$$|f_a| \leq |p^{(1)}| + \eta^2, \quad |g_a| \leq 1, \quad (3.14)$$

for an appropriate complex domain containing the real interval where we consider the motion of ϕ_a , $p_a^{(1)}$, N . In this section, in all the inequalities we omit all the finite constants

and $|p^{(1)}|$ is interpreted as the quantity bounded by $M|p^{(1)}|$ for some positive constant M and

$$|p^{(1)}| := \sum_a |p_a^{(1)}|. \quad (3.15)$$

From now on, we will simply write $p_a^{(1)}$ as p_a for notational simplicity. In this section, we consider the evolution for $0 \leq N \leq 1/\eta^2$ during which the scalar fields roll slowly on the potential U . For a function $f(N)$, let us define $\|f\|$ by

$$\|f\| := \sup_{0 \leq N \leq 1/\eta^2} |f(N)|. \quad (3.16)$$

Under these notations, following propositions hold.

Proposition 4 *Let k nonnegative integer, and δ_c small positive constant. Under the initial conditions*

$$\frac{\partial^k}{\partial \lambda^k} \phi(0), \quad \frac{\partial^k}{\partial \lambda^k} p(0), \quad \frac{\partial^k}{\partial \lambda^k} c_T \sim \delta_c^k, \quad (3.17)$$

for $0 \leq N \leq 1/\eta^2$, the upper bounds of the independent variables are given by

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi \right| \leq \delta_c^k, \quad (3.18)$$

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p \right| \leq \delta_c^k (e^{-3N} + \eta^2), \quad (3.19)$$

For the proof, see Appendix F.

In the above and from now on, as for λ differentiations of the physical quantities at the initial time $N = 0$, the suffixes a implying “ a fixed” are omitted since what the λ differentiations operate on do not contain any $a = e^N$ dependent parts.

Proposition 5 *Let k nonnegative integer. The differences $\Delta\phi_a := \phi_a - \bar{\phi}_a$ where ϕ obey the exact evolution equations (3.11) (3.12) and $\bar{\phi}$ obey the truncated evolution equations as*

$$\frac{d}{dN} \bar{\phi} = \eta^2 F(\bar{\phi}), \quad (3.20)$$

are bounded as

$$\left\| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \Delta\phi \right\| \leq \eta^2 \delta_c^k \sim \eta^2 \left\| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi \right\|, \quad (3.21)$$

under the initial conditions

$$\frac{\partial^k}{\partial \lambda^k} \Delta\phi(0) = 0. \quad (3.22)$$

For the proof, see Appendix G.

According to Proposition 5, if we want to investigate the evolution of the leading order in the slow rolling (η^2) expansion, we only have to solve rather simple evolution equations (3.20). In the following subsections, we study the evolutionary behaviors of the above evolution equations (3.20).

3.2 the τ function and the N potential

In this subsection, we introduce the τ function and the N potential which enable us to trace the multiple slow rolling scalar fields in the long time interval analytically. By using the τ function as the evolution parameter, the evolutions of the scalar fields ϕ_a and the old evolution parameter N can be expressed in the form of the simple analytic function of τ . From these expressions of ϕ_a in terms of the τ function, we can calculate the N potential in the simple way. The N potential is written in terms of the initial values of the scalar fields $\phi_a(0)$ only. It not only represents the difference between the e-folding number of the scale factor at the end of the slow rolling phase and that at the first horizon crossing, but also has the complete information as to the nonlinear curvature perturbations. The λ differentiations of the N potential generate the S formulae connecting the amplitudes of all the higher order Bardeen parameters at the end of the slow rolling phase with the scalar fields perturbations at the first horizon crossing.

Under the conditions (3.1), we will investigate the evolutionary behaviors of the solution of the evolution equations which are obtained by the truncation explained in the previous subsection.

$$\frac{d}{dN}\phi_a = -\frac{1}{\kappa^2} \frac{1}{U} \frac{\partial U}{\partial \phi_a}. \quad (3.23)$$

From the present subsections, we will call off the nondimensionalization in the previous subsection, since the truncated evolution equations (3.23) have already been made sufficiently simple. In the multiple scalar fields case, the evolution equations (3.23) cannot be solved analytically, and the scalar fields ϕ_a cannot be expressed in the form of the well known function of N . So by replacing the old evolution parameter N with the new evolution parameter τ , we decompose (3.23) into two parts:

$$\frac{d}{d\tau}\phi_a = -\frac{\partial U}{\partial \phi_a}, \quad (3.24)$$

$$\frac{d}{d\tau}N = \kappa^2 U. \quad (3.25)$$

The new evolution parameter τ introduced in the above equations will be called the τ function from now on. By introducing the τ function, the evolution equations become simple enough to be solved analytically. In the potential (3.1), we can easily solve (3.24) by iteration, then we can get $\phi_a(\tau)$ expressed in the form of the analytic functions of the τ function. By substituting these $\phi_a(\tau)$ into U in (3.25), and integrating (3.25) with respect to the τ function, we obtain the expression of the old evolution parameter N in terms of the new evolution parameter τ :

$$N = \int_0^\tau d\tau \kappa^2 U. \quad (3.26)$$

The expressions of ϕ_a , N in terms of the τ function describe the dynamical evolutions of our multiple slow rolling scalar fields system completely.

From this subsection, we adopt the τ function as the evolution parameter. We introduce $(\partial/\partial\lambda)_\tau$ as the λ differentiation taken at the fixed τ , that is, operating on the locally homogeneous quantities expressed by using the τ function as the evolution parameter. In order to exaggerate the fact that the τ derivative and $(\partial/\partial\lambda)_\tau$ are commutative, we use

$(\partial/\partial\tau)_\tau$ as the τ derivative from now on. By using $(\partial/\partial\lambda)_\tau$, $(\partial/\partial\tau)_\tau$, D defined by (2.72) can be expressed as

$$D = \left(\frac{\partial}{\partial\lambda}\right)_\tau - \left\{ \left(\frac{\partial a}{\partial\lambda}\right)_\tau / \left(\frac{\partial a}{\partial\tau}\right)_\tau \right\} \left(\frac{\partial}{\partial\tau}\right)_\tau, \quad (3.27)$$

and $\bar{\delta}$ defined by (2.76) can be expressed as

$$\bar{\delta} = \left(\frac{\partial}{\partial\lambda}\right)_\tau - \left\{ \left(\frac{\partial U}{\partial\lambda}\right)_\tau / \left(\frac{\partial U}{\partial\tau}\right)_\tau \right\} \left(\frac{\partial}{\partial\tau}\right)_\tau, \quad (3.28)$$

where we used

$$\frac{d}{d\lambda} = \left(\frac{\partial}{\partial\lambda}\right)_\tau + \frac{d\tau}{d\lambda} \left(\frac{\partial}{\partial\tau}\right)_\tau, \quad (3.29)$$

and $\rho = U$ which holds under the present truncation (3.23). Then by using the τ function as the evolution parameter, the Bardeen parameter ζ_n defined by (2.75) can be decomposed as

$$\zeta_n = \frac{\partial^n}{\partial\lambda^n} \bar{N} - \frac{\kappa^2}{4} \bar{\delta}^n A(0, 0). \quad (3.30)$$

By \bar{N} , we represent

$$\bar{N} := \int_0^\infty d\tau \kappa^2 U, \quad (3.31)$$

and this \bar{N} will be called the N potential from now on. In the above and from now on, as for λ differentiations of the physical quantities at the initial time $\tau = 0$, the suffixes τ implying “ τ fixed” are omitted since what the λ differentiations operate on do not contain any τ dependent parts. $A(2n, k)$ is defined by

$$A(0, 0) := 4 \int_\tau^\infty d\tau U, \quad (3.32)$$

$$A(2n, k) := \left(-\frac{1}{2}\right)^n \left(\frac{\partial^n}{\partial\tau^n} \frac{\partial^k}{\partial\lambda^k}\right)_\tau A(0, 0). \quad (3.33)$$

By using $A(2n, k)$, $\bar{\delta}$ can be expressed as

$$\bar{\delta} = \left(\frac{\partial}{\partial\lambda}\right)_\tau + \frac{1}{2} \frac{A(2, 1)}{A(4, 0)} \left(\frac{\partial}{\partial\tau}\right)_\tau. \quad (3.34)$$

Afterward we will prove the fact that as for the Bardeen parameter ζ_n at the end of the slow rolling phase the first term in (3.30) has leading contribution in the slow rolling expansion scheme. In order to prove this statement, we put forth several assumptions. All the investigations in this section will be established under the following assumptions:

- (i) All the masses of the scalar fields are of the same order.

$$m_a^2 \sim m^2, \quad (3.35)$$

where

$$m^2 := \min_a \{m_a^2\}. \quad (3.36)$$

- (ii) The interaction potential U_{int} is the sum of the m -th order monomials ($m \geq 3$) and satisfies

$$|U_{\text{int}}| \leq \mu_c m^2 \frac{1}{\phi_0} |\phi|^3, \quad (3.37)$$

for $|\phi| \leq \phi_0$. ϕ_0 is defined by the positive constant of the order of

$$\phi_0 \sim \phi_a(0), \quad (3.38)$$

and $|\phi|$ is defined by

$$|\phi|^2 := \sum_a |\phi_a|^2, \quad (3.39)$$

and μ_c is the small positive constant characterizing the interaction strength.

- (iii) The n -th order perturbation at the initial time $\tau = 0$ is of the order of

$$\frac{\partial^k}{\partial \lambda^k} \phi_a(0) \sim \delta_c^k \phi_0, \quad (3.40)$$

where δ_c is the small constant characterizing the perturbation size.

Under these assumptions, the following proposition holds.

Proposition 6 *The following estimations hold.*

$$\frac{\partial^k}{\partial \lambda^k} \bar{N} = \frac{\kappa^2}{4} \sum_a \frac{\partial^k}{\partial \lambda^k} [\phi_a^2(0)] + < \kappa^2 \mu_c \delta_c^k \phi_0^2 >, \quad (3.41)$$

$$A(2n, k) = \sum_a (m_a^2)^n \frac{\partial^k}{\partial \lambda^k} [\phi_a^2(0)] \exp[-2m_a^2 \tau] + < \mu_c m^{2n} \delta_c^k \phi_0^2 \exp[-2m^2 \tau] >, \quad (3.42)$$

where $< M >$ is the quantity bounded by M .

For the proof, see Appendix H.

At the end of the slow rolling phase when the Hubble parameter is of the order of the scalar fields mass, the values of the scalar fields are of the order of the Planck mass $\phi_a(\tau) \sim 1/\kappa$. According to Proposition 6, at the end of the slow rolling phase, we obtain the estimations as

$$\frac{\partial^k}{\partial \lambda^k} \bar{N} \sim \kappa^2 \delta_c^k \phi_0^2, \quad (3.43)$$

$$A(2n, k) \sim m^{2n} \delta_c^k \phi_0^2 \left(\frac{m}{H_0} \right)^2, \quad (3.44)$$

where H_0 is the Hubble parameter at the initial time $\tau = 0$. By using the above estimations, the second term of (3.30) can be estimated as

$$-\frac{\kappa^2}{4} \bar{\delta}^n A(0, 0) \sim \kappa^2 \delta_c^n \phi_0^2 \left(\frac{m}{H_0} \right)^2 \sim \frac{\partial^n}{\partial \lambda^n} \bar{N} \cdot \left(\frac{m}{H_0} \right)^2. \quad (3.45)$$

Since the second term of (3.30) is suppressed by the slow rolling parameter $(m/H_0)^2$ compared to the first term, the main part of the nonlinear n -th order Bardeen parameter ζ_n at the end of the slow rolling phase is given by the λ derivative of the N potential:

$$\zeta_n = \frac{\partial^n}{\partial \lambda^n} \bar{N}. \quad (3.46)$$

So in order to obtain the final amplitude of the Bardeen parameter ζ_n , we have only to calculate the N potential \bar{N} .

In particular case, the N potential can be calculated very easily. When the scalar fields potential is written in the separable form

$$U = \sum_a U_a(\phi_a), \quad (3.47)$$

the N potential can also be expressed in the separable form as

$$\bar{N} = \sum_a \bar{N}(a), \quad (3.48)$$

where

$$\bar{N}(a) := \kappa^2 \int_0^{\phi_a(0)} d\phi_a \left(U_a / \frac{\partial U_a}{\partial \phi_a} \right). \quad (3.49)$$

3.3 Exactly solvable model; noninteracting case

In this subsection, in the multiple free fields case, we give the Bardeen parameter and the entropic perturbations and the gravitational wave perturbations and their spectral indices. Since we can obtain the analytic expression of $\phi_a(\tau)$, all the results obtained in this subsection have no black boxes originating from the S formulae connecting the amplitudes of the final physical quantities and those of the initial physical quantities, which were unknown in the paper [4].

We consider the exactly solvable model defined by

$$U = \sum_a \frac{1}{2} m_a^2 \phi_a^2, \quad (3.50)$$

and calculate the various physical quantities. The N potential is calculated as

$$\bar{N} = \frac{\kappa^2}{4} \sum_a \phi_a^2(0), \quad (3.51)$$

then the Bardeen parameter ζ_n at the end of the slow rolling phase is calculated by (3.46). From now on, we consider the linear perturbations, and calculate their spectral indices. We consider the entropy perturbation between ϕ_a and ϕ_b defined by

$$S_{ab} := -3H \left(\frac{1}{\dot{\rho}_a} D\rho_a - \frac{1}{\dot{\rho}_b} D\rho_b \right). \quad (3.52)$$

In the present case (3.50), S_{ab} is given by

$$S_{ab} = 3\kappa^2 U \left(\frac{1}{m_a^2 \phi_a(0)} \frac{\partial \phi_a(0)}{\partial \lambda} - \frac{1}{m_b^2 \phi_b(0)} \frac{\partial \phi_b(0)}{\partial \lambda} \right). \quad (3.53)$$

In order to calculate spectral indices, we have to know the amplitudes of the quantum fluctuations of the scalar fields and the gravitational waves at the first horizon crossing $\tau = 0$:

$$\left\langle \left\langle \frac{\partial \phi_a(0)}{\partial \lambda} \frac{\partial \phi_b(0)}{\partial \lambda} \right\rangle \right\rangle \sim H^2 \delta_{ab} \sim \kappa^2 U \delta_{ab}, \quad (3.54)$$

$$\frac{1}{\kappa^2} \left\langle \left\langle \frac{\partial \tilde{\gamma}_{ij}(0)}{\partial \lambda} \frac{\partial \tilde{\gamma}_{ij}(0)}{\partial \lambda} \right\rangle \right\rangle \sim H^2 \sim \kappa^2 U. \quad (3.55)$$

When we calculate the above correlation functions, for simplicity we do not take into account the first order slow rolling corrections unlike the paper [4], so the above correlation functions become diagonal. From the horizon crossing relation as

$$k = aH = \frac{\kappa}{\sqrt{3}} e^N U^{1/2}, \quad (3.56)$$

we obtain

$$\begin{aligned} d \ln k &= \left(\kappa^2 U + \frac{1}{2U} \frac{dU}{d\tau} \right) d\tau \\ &= \kappa^2 U \left\{ 1 + O\left(\frac{m^2}{H_0^2}\right) \right\} d\tau \\ &\approx \kappa^2 U d\tau. \end{aligned} \quad (3.57)$$

Then we can calculate the spectral indices as

$$\frac{\partial}{\partial \ln k} \ln \langle \zeta_1^2 \rangle = -\frac{4}{\kappa^2} \frac{1}{B(2)} \left(\frac{B(2)}{B(0)} + \frac{B(4)}{B(2)} \right), \quad (3.58)$$

$$\frac{\partial}{\partial \ln k} \ln \langle S_{ab}^2 \rangle = \frac{4}{\kappa^2} \frac{1}{B(2)} \left(m_a^2 m_b^2 \frac{m_a^2 \phi_a^2(0) + m_b^2 \phi_b^2(0)}{m_a^4 \phi_a^2(0) + m_b^4 \phi_b^2(0)} - \frac{B(4)}{B(2)} \right), \quad (3.59)$$

$$\frac{\partial}{\partial \ln k} \ln \langle \tilde{\gamma}_{ij}^2 \rangle = -\frac{4}{\kappa^2} \frac{B(4)}{B(2)^2}, \quad (3.60)$$

where

$$B(2n) := \sum_a (m_a^2)^n \phi_a^2(0). \quad (3.61)$$

As for the correlation between the adiabatic and the entropic perturbations defined by

$$\mathcal{C}_{ab} := \frac{\langle \zeta_1 S_{ab} \rangle}{\sqrt{\langle \zeta_1^2 \rangle} \sqrt{\langle S_{ab}^2 \rangle}}, \quad (3.62)$$

we obtain

$$\frac{\partial}{\partial \ln k} \ln \mathcal{C}_{ab} = \frac{2}{\kappa^2} \frac{1}{B(0)} - \frac{2}{\kappa^2} \frac{1}{B(2)} m_a^2 m_b^2 \frac{m_a^2 \phi_a^2(0) + m_b^2 \phi_b^2(0)}{m_a^4 \phi_a^2(0) + m_b^4 \phi_b^2(0)}. \quad (3.63)$$

Since we succeed in solving the scalar fields evolutions completely, our expressions of the spectral indices of the perturbation variables are expressed in terms of the initial fields values and the masses of the scalar fields only, unlike the paper [4] where the author did not solve the scalar fields evolutions and their expressions of the spectral indices contain unknown factors originating from the scalar fields evolutions.

3.4 Effect of the nonresonant interactions

In the previous subsection, we considered the free scalar fields. In this subsection, we will consider the interacting case. All the interacting systems are classified into the nonresonant cases and the resonant cases. By the word “resonance”, we mean all the factors inducing the small denominators in the perturbative expansion [1]. Therefore the resonance phenomena are not confined in the oscillatory dynamical system. In fact, the resonance can also occur in the present slow rolling system. In the nonresonant case, the masses of the scalar fields do not satisfy any resonant relations as shown in Proposition 7 presented in the below. In the resonant case, the masses of the scalar fields satisfy more than one resonant relation. In the nonresonant case, the Poincaré theorem greatly simplifies the treatment of the interactions. In the nonresonant case, all the interaction terms can be removed from the evolution equations by performing the suitable transformation of the field variables. This process is called the linearization of the evolution equations. Since the linearized, that is transformed evolution equations are the free fields equations, they can be solved easily. The effects of all the interaction terms are contained in the transformation law of the field variables. The fact that the masses of the scalar fields do not satisfy any resonant relations guarantees the convergence of the transformation law of the field variables. In this way, the evolutions of the interacting scalar fields ϕ_a can be solved completely in the form of the analytic functions of the τ function. As for the concrete nonresonant model, we determine the evolutions of the scalar fields $\phi_a(\tau)$ and the N potential. By using this N potential, we can calculate the nonlinearity parameters f_{NL} , g_{NL} representing the non-Gaussianity of the Bardeen parameter. These nonlinearity parameters f_{NL} , g_{NL} are shown to be suppressed by the slow rolling parameter. Therefore it is difficult to observe such small non-Gaussianity.

First we present the proposition about the nonresonant condition. In order to express the nonresonant condition, we introduce several notations. By α , k , we represent

$$\alpha := (m_1^2, m_2^2, \dots, m_{N_S}^2), \quad (3.64)$$

$$k := (k_1, k_2, \dots, k_{N_S}), \quad (3.65)$$

where all k_a belong to the nonnegative integer set \mathbb{Z}_{0+} :

$$k_a \in \mathbb{Z}_{0+}, \quad (3.66)$$

$$\mathbb{Z}_{0+} := \{k \in \mathbb{Z} | k \geq 0\}. \quad (3.67)$$

By $|k|$, we represent

$$|k| := \sum_a |k_a|. \quad (3.68)$$

Then the following proposition holds.

Proposition 7 Suppose that for an arbitrary $(a; k) \in \{1, 2, \dots, N_S\} \times \mathbb{Z}_{0+}^N$ satisfying $|k| \geq 2$,

$$(k \cdot \alpha) - \alpha_a \neq 0 \quad (3.69)$$

holds and for an arbitrary $a \in \{1, 2, \dots, N_S\}$,

$$m_a^2 > 0 \quad (3.70)$$

holds. Then there exists a positive constant δ_m satisfying

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_m, \quad (3.71)$$

for an arbitrary $(a; k) \in \{1, 2, \dots, N_S\} \times \mathbb{Z}_{0+}^N$ satisfying $|k| \geq 2$.

For the proof, see Appendix I.

Under the condition of Proposition 7, we can prove the following proposition.

Proposition 8 We consider the evolution equation

$$\frac{\partial}{\partial \tau} \phi_a = -\alpha_a \phi_a + \tilde{f}_a(\phi), \quad (3.72)$$

where \tilde{f}_a is the sum of m -th order monomials of ϕ_a ($m \geq 2$) satisfying

$$|\tilde{f}_a(\phi)| < \mu_c \frac{\alpha_M}{\phi_0} |\phi|^2, \quad (3.73)$$

where α_M is the maximum of α_a , for $|\phi| \leq \phi_0$. Under the condition that α satisfies (3.71), there exists a positive constant R such that, for φ satisfying $|\varphi| \leq R$, there exists the transformation

$$\phi_a = \varphi_a + w_a(\varphi), \quad (3.74)$$

where w_a is the sum of m -th order monomials of φ_a ($m \geq 2$) and satisfies

$$|w_a(\varphi)| \leq \frac{\mu_c}{\phi_0} |\varphi|^2. \quad (3.75)$$

By this transformation law (3.74), (3.72) is transformed into the linear differential equation as

$$\frac{\partial}{\partial \tau} \varphi_a = -\alpha_a \varphi_a. \quad (3.76)$$

Proposition 8 is well known as the Poincaré theorem. [1] The proof is given in Appendix J. This theorem can be applied only to the differential equations containing the linear terms plus the small perturbations. Since they are not even the Hamiltonian dynamical systems in general, Proposition 8 does not cover the anharmonic oscillator. It is well known that the evolution equation of the nonlinear anharmonic oscillator cannot be transformed into the linear evolution equation. But for the present purpose of treating the multiple slow rolling scalar fields systems, Proposition 8 is enough. All the terms in $w_a(\varphi)$ in (3.74) have factors as

$$\frac{1}{(k \cdot \alpha) - \alpha_a}. \quad (3.77)$$

In order to prove the convergence of this transformation law (3.74) by the majorant method, the nonresonant inequalities (3.71) are essential. According to Proposition 8, by (3.74) and

$$\varphi_a = \varphi_a(0) \exp[-\alpha_a \tau], \quad (3.78)$$

the evolutions of the scalar fields ϕ_a can be expressed as an analytic function of τ .

Next we consider the concrete example whose potential U is given by

$$U = \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2 + \frac{g}{4}\phi_1^2\phi_2^2, \quad (3.79)$$

where m_1^2, m_2^2 are assumed to be nonresonant as in Proposition 7. We obtain

$$\begin{aligned} \phi_1 &= \phi_1(0) \exp(-m_1^2\tau) - \frac{g}{4m_2^2}\phi_1(0)\phi_2^2(0) [\exp(-m_1^2\tau) - \exp\{-(m_1^2 + 2m_2^2)\tau\}] \\ &+ \dots, \end{aligned} \quad (3.80)$$

$$\begin{aligned} \phi_2 &= \phi_2(0) \exp(-m_2^2\tau) - \frac{g}{4m_1^2}\phi_2(0)\phi_1^2(0) [\exp(-m_2^2\tau) - \exp\{-(m_2^2 + 2m_1^2)\tau\}] \\ &+ \dots. \end{aligned} \quad (3.81)$$

Then we obtain the N potential as

$$\frac{1}{\kappa^2}\bar{N} = \frac{1}{4}\phi_1^2(0) + \frac{1}{4}\phi_2^2(0) - \frac{g}{8(m_1^2 + m_2^2)}\phi_1^2(0)\phi_2^2(0) + \dots. \quad (3.82)$$

In this model, we consider the nonlinear parameters f_{NL}, g_{NL} [14] defined by

$$f_{NL} = \frac{5}{6} \frac{\bar{N}_{ab}\bar{N}^a\bar{N}^b}{(\bar{N}_a\bar{N}^a)^2}, \quad (3.83)$$

$$g_{NL} = \frac{25}{54} \frac{\bar{N}_{abc}\bar{N}^a\bar{N}^b\bar{N}^c}{(\bar{N}_a\bar{N}^a)^3}, \quad (3.84)$$

where

$$\bar{N}_a := \frac{\partial}{\partial\phi_a(0)}\bar{N}, \quad \bar{N}_{ab} := \frac{\partial^2}{\partial\phi_a(0)\partial\phi_b(0)}\bar{N}, \dots. \quad (3.85)$$

We obtain

$$f_{NL} = \frac{5}{12} \frac{1}{\bar{N}} \sim \left(\frac{m}{H_0}\right)^2, \quad (3.86)$$

$$g_{NL} = -\frac{25}{144} \frac{g\kappa^2}{m_1^2 + m_2^2} \phi_1^2(0)\phi_2^2(0) \frac{1}{\bar{N}^3} \sim \frac{U_{\text{int}}}{U} \left(\frac{m}{H_0}\right)^4. \quad (3.87)$$

So we can conclude that the nonlinear parameters f_{NL}, g_{NL} are suppressed by the slow rolling parameter $(m/H_0)^2$. For an arbitrary potential model satisfying the assumptions (i), (ii) and (iii) presented above Proposition 6 and for the nonresonant masses as explained in Proposition 7, the N potential can be written as

$$\frac{1}{\kappa^2}\bar{N} = \frac{1}{4} \sum_a \phi_a^2(0) + \tilde{g}(\phi(0)), \quad (3.88)$$

where \tilde{g} is the sum of m -th order monomials of $\phi_a(0)$ ($m \geq 3$) and satisfies

$$\tilde{g}(\phi(0)) \leq \mu_c \frac{1}{\phi_0} |\phi(0)|^3, \quad (3.89)$$

for $|\phi(0)| \leq \phi_0$. Then we obtain the nonlinear parameters as

$$f_{NL} = \frac{5}{12} \frac{1}{\bar{N}} \sim \left(\frac{m}{H_0} \right)^2, \quad g_{NL} \sim \mu_c \left(\frac{m}{H_0} \right)^4, \quad (3.90)$$

that is, the nonlinear parameters f_{NL} , g_{NL} are suppressed by the slow rolling parameters $(m/H_0)^2$.

3.5 Effect of the resonant interactions

According to Proposition 8 in the previous subsection, all the nonresonant interaction terms can be eliminated by the field transformations $\phi_a \rightarrow \varphi_a$. But in the resonant interaction case, such linearization cannot be applied. In this subsection, we calculate the effect of the resonant interaction by the iteration method. Perturbatively at least, the resonant interactions do not generate any special effects in the N potential.

As the concrete example, we consider the model defined by

$$U_{\text{int}} = \lambda \phi_1 \phi_2^2, \quad m_1^2 = 2m_2^2. \quad (3.91)$$

It can be solved as

$$\phi_1 = \phi_1(0) \exp(-m_1^2 \tau) - \lambda \phi_2^2(0) \tau \exp(-m_1^2 \tau) + \dots, \quad (3.92)$$

$$\begin{aligned} \phi_2 &= \phi_2(0) \exp(-m_2^2 \tau) - \frac{2\lambda}{m_1^2} \phi_1(0) \phi_2(0) [\exp(-m_2^2 \tau) - \exp\{-(m_1^2 + m_2^2)\tau\}] \\ &+ \dots \end{aligned} \quad (3.93)$$

where in the right hand side of ϕ_1 the second term proportional to $\lambda\tau$ appears because of the resonant interaction. The N potential is given by

$$\frac{1}{\kappa^2} \bar{N} = \frac{1}{4} \phi_1^2(0) + \frac{1}{4} \phi_2^2(0) - \frac{5\lambda}{8m_2^2} \phi_1(0) \phi_2^2(0) + \dots \quad (3.94)$$

So the resonant interaction terms generate no singular terms in the N potential \bar{N} .

§4 Discussion

We presented the nonlinear LWL formula and by using it we investigated the evolutionary behaviors of the nonlinear cosmological perturbations on superhorizon scales in the universe dominated by the multiple slow rolling scalar fields.

In the excellent study as to the multiple slow rolling scalar fields [5] [4], the multiple scalar fields were decomposed into the adiabatic field and the entropy fields instant by instant. By using this decomposition, it was pointed out that the growing of the Bardeen

parameter corresponds to the curvature of the trajectory in the scalar fields space. Further in the paper [4], the spectral index (the wavenumber dependences) of the Bardeen parameter was presented by calculating the differential coefficients at the instant of the first horizon crossing. But this study is the local investigation because the author did not calculate the S formulae connecting the final amplitudes of the adiabatic and the entropic fields variables at the end of the slow rolling phase with the initial amplitudes of them at the first horizon crossing, and they calculated only the time derivative of the S formulae at the first horizon crossing by using the perturbation evolution equations. So the formulae of the spectral indices in the paper [4] have black boxes (unknown factors) which come from the S formulae which could not be determined. Therefore as the method for investigating the evolutions of the scalar fields perturbations in the long time interval such as in the whole slow rolling phase analytically, we present the method using the τ function and the N potential. This method enables us to calculate the S formulae analytically. In fact, our expressions of spectral indices in subsection 3.3 can be expressed in terms of the initial field values and the masses of the scalar fields only and they have no blackboxes since we succeed in calculating the scalar fields evolutions containing the S formulae completely. Our method will deepen the understanding of the dynamical evolutions of the multiple slow rolling scalar fields in the long time interval with high accuracy.

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§A Derivation of (2.59)

In this appendix, we rewrite the momentum constraints (2.17) into the constraints which the solution constants of the solution of the locally homogeneous universe must satisfy.

By using (2.9)(2.51), the left hand side of the momentum constraint (2.17) is expressed as

$$\begin{aligned} & D_i \tilde{A}^i_j - \frac{2}{3} D_j K \\ &= \frac{1}{\alpha} \left[\frac{1}{2} \frac{\partial_i \alpha}{\alpha} \tilde{\gamma}^{ik} \partial_t \tilde{\gamma}_{kj} + \frac{1}{2} \tilde{\gamma}^{il} \partial_i \tilde{\gamma}_{lm} \tilde{\gamma}^{mk} \partial_t \tilde{\gamma}_{kj} - \frac{1}{2} \tilde{\gamma}^{ik} \partial_i \partial_t \tilde{\gamma}_{kj} \right. \\ & \quad \left. - \frac{3}{2} \frac{\partial_k a}{a} \tilde{\gamma}^{kl} \partial_t \tilde{\gamma}_{lj} + \frac{1}{2} {}^s \tilde{\Gamma}_{ij}^k \tilde{\gamma}^{il} \partial_t \tilde{\gamma}_{lk} + 2\alpha \partial_j \left(\frac{H}{\alpha} \right) \right], \end{aligned} \quad (\text{A.1})$$

where ${}^s \tilde{\Gamma}_{ij}^k$ is the Christoffel symbol of $\tilde{\gamma}_{ij}$. As for the right hand side of (A.1),

$$\frac{1}{2} \frac{\partial_i \alpha}{\alpha} \tilde{\gamma}^{ik} \partial_t \tilde{\gamma}_{kj} - \frac{3}{2} \frac{\partial_k a}{a} \tilde{\gamma}^{kl} \partial_t \tilde{\gamma}_{lj} = -\frac{1}{2} \frac{\alpha}{a^3} \partial_i \left(\frac{a^3}{\alpha} \right) \left(M^{-1} \dot{M} \right)^i_j, \quad (\text{A.2})$$

and

$${}^s \tilde{\Gamma}_{ij}^k \tilde{\gamma}^{il} \partial_t \tilde{\gamma}_{lk} = \frac{1}{2} \text{tr} \left[M^{-1} \partial_j M M^{-1} \dot{M} \right], \quad (\text{A.3})$$

and

$$\tilde{\gamma}^{il}\partial_i\tilde{\gamma}_{lm}\tilde{\gamma}^{mk}\partial_t\tilde{\gamma}_{kj} = \left(M^{-1}\partial_i M M^{-1}\dot{M}\right)_j^i, \quad (\text{A.4})$$

and

$$\tilde{\gamma}^{ik}\partial_i\partial_t\tilde{\gamma}_{kj} = \left(M^{-1}\partial_i\dot{M}\right)_j^i. \quad (\text{A.5})$$

Therefore we obtain

$$\begin{aligned} & D_i\tilde{A}_j^i - \frac{2}{3}D_j K \\ &= \frac{1}{\alpha} \left[-\frac{1}{2}\frac{\alpha}{a^3}\partial_i T_j^i + \frac{1}{4}\frac{\alpha}{a^3}\text{tr}(R^{-1}\partial_j R T) \right. \\ & \quad \left. + \frac{\alpha}{a^3}\int_{t_0} dt \partial_j \left(\frac{\alpha}{a^3}\right) c_T + \frac{1}{2}\frac{\alpha}{a^3}\int_{t_0} dt \frac{\alpha}{a^3}\partial_j c_T + 2\alpha\partial_j \left(\frac{H}{\alpha}\right) \right], \end{aligned} \quad (\text{A.6})$$

where we used the solution of M (2.53).

The right hand side of (2.17) is expressed as

$$\kappa^2 J_j = \frac{1}{\alpha}[\kappa^2 h(v_j + \beta_j)]. \quad (\text{A.7})$$

By summing (2.44) with fluid indices α , we obtain

$$[h(\beta_i + v_i)]_f = \frac{\alpha}{a^3} \sum_{\alpha} C_{\alpha i} + \frac{\alpha}{a^3} \int_{t_0} dt \alpha a^3 \left[-\partial_i P_f - \frac{1}{\alpha} D_i \alpha h_f - \sum_a S_a \partial_i \phi_a \right] + O(\epsilon^3), \quad (\text{A.8})$$

where we used $\sum_{\alpha} Q_{\alpha i} = -(Q_i)_S = -\sum_a S_a \partial_i \phi_a$ and as for the scalar fields components we obtain

$$[h_a(\beta_i + v_{ai})]_S = -\sum_a \dot{\phi}_a \partial_i \phi_a + O(\epsilon^3). \quad (\text{A.9})$$

We substitute the sum of (A.8) and (A.9) into $h(\beta_i + v_i)$ in (A.7).

Then through simple calculations we obtain (2.59).

§B Lie derivative in the leading order of the gradient expansion

The Lie derivatives of the quantity of upper index and the quantity with lower index are expressed as

$$L(T)X^\mu = T^\alpha \partial_\alpha X^\mu - \partial_\alpha T^\mu X^\alpha, \quad (\text{B.1})$$

$$L(T)X_\mu = T^\alpha \partial_\alpha X_\mu + X_\alpha \partial_\mu T^\alpha, \quad (\text{B.2})$$

respectively. By the Leibniz rule, these definitions are expanded into the tensor of an arbitrary rank. For example, the Lie derivative of the metric is given by

$$L(T)g_{\mu\nu} = T^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu T^\alpha + g_{\mu\alpha} \partial_\nu T^\alpha. \quad (\text{B.3})$$

From now on, we consider the gradient expansion scheme defined by

$$\partial_i = O(\epsilon), \quad T^i = O(\epsilon), \quad g_{i0} = O(\epsilon), \quad (\text{B.4})$$

and the $O(\epsilon^2)$ order corrections are dropped. For an arbitrary scalar quantity A , in our scheme (B.4) the Lie derivative of A is written by

$$L(T)A = T^0 \frac{d}{dt} A + O(\epsilon^2). \quad (\text{B.5})$$

By using (B.3) to $g_{00} = -\alpha^2 + \beta_k \beta^k$ and $g_{ij} = \gamma_{ij}$, we obtain

$$L(T)\alpha = T^0 \frac{d}{dt} \alpha + \alpha \frac{d}{dt} T^0 + O(\epsilon^2), \quad (\text{B.6})$$

$$L(T)\gamma_{ij} = T^0 \frac{d}{dt} \gamma_{ij} + O(\epsilon^2). \quad (\text{B.7})$$

From (B.5) and (B.6), we obtain

$$L(T) \left(\frac{\dot{A}}{\alpha} \right) = T^0 \frac{d}{dt} \left(\frac{\dot{A}}{\alpha} \right) + O(\epsilon^2). \quad (\text{B.8})$$

From this equation, we can see that the Lie derivative of \dot{A}/α where A is a scalar quantity has the same form as the Lie derivative of the scalar quantity (B.5). By using (B.7) to $\det(\gamma_{ij}) = a^6$, we obtain

$$L(T)a = T^0 \frac{d}{dt} a + O(\epsilon^2), \quad (\text{B.9})$$

therefore we obtain

$$L(T)\tilde{\gamma}_{ij} = T^0 \frac{d}{dt} \tilde{\gamma}_{ij} + O(\epsilon^2). \quad (\text{B.10})$$

In our gradient expansion scheme, the scale factor a and $\tilde{\gamma}_{ij}$ can be regarded as the scalar quantity.

§C Proof of Proposition 1

The gauge transformation is described by the differential equation (2.70). So we can consider the μ derivative as the gauge transformation. By differentiating (2.70) with respect to λ , we obtain

$$\frac{d}{d\mu} \frac{dA}{d\lambda} = L \left(\frac{dT}{d\lambda} \right) A + L(T) \frac{dA}{d\lambda}, \quad (\text{C.1})$$

$$\frac{d}{d\mu} \frac{d^2 A}{d\lambda^2} = L \left(\frac{d^2 T}{d\lambda^2} \right) A + 2L \left(\frac{dT}{d\lambda} \right) \frac{dA}{d\lambda} + L(T) \frac{d^2 A}{d\lambda^2}, \quad (\text{C.2})$$

and so on. As for $D^n A$ where D is defined by (2.72) and A is an arbitrary scalar quantity, we can prove

$$\frac{d}{d\mu} D^n A = L(T) D^n A, \quad (\text{C.3})$$

for an arbitrary natural number n .

Proof

If we interpret $D^0 A := A$, (C.3) holds evidently for $n = 0$. We assume that (C.3) holds for $n = k - 1$ where $k = 1, 2, \dots$. Then for $n = k$

$$\begin{aligned}
\frac{d}{d\mu} D^k A &= \frac{d}{d\mu} \left\{ \left(\frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt} \right) D^{k-1} A \right\} \\
&= \frac{d}{d\lambda} \frac{d}{d\mu} D^{k-1} A - \frac{d}{d\lambda} \left(\frac{da}{d\mu} \right) \frac{1}{\dot{a}} (D^{k-1} A)^\cdot - \frac{da}{d\lambda} \frac{d}{d\mu} \left\{ \frac{1}{\dot{a}} (D^{k-1} A)^\cdot \right\} \\
&= L \left(\frac{dT}{d\lambda} \right) D^{k-1} A + L(T) \frac{d}{d\lambda} D^{k-1} A - \left\{ L \left(\frac{dT}{d\lambda} \right) a + L(T) \frac{da}{d\lambda} \right\} \frac{1}{\dot{a}} (D^{k-1} A)^\cdot \\
&\quad - \frac{da}{d\lambda} L(T) \left\{ \frac{1}{\dot{a}} (D^{k-1} A)^\cdot \right\} \\
&= L(T) \left\{ \left(\frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt} \right) D^{k-1} A \right\} \\
&= L(T) D^k A
\end{aligned} \tag{C.4}$$

where we use the fact that both $D^{k-1} A$ and the scale factor a have the Lie derivatives of the scalar quantity type (B.5). For $n = k$, (C.3) holds. By induction, we complete the proof. ■

By putting $\lambda = 0$ in (C.3) and by noticing $T(\lambda = 0) = 0$

$$\left. \frac{d}{d\mu} D^n A \right|_{\lambda=0} = 0. \tag{C.5}$$

Then we proved that $D^n A$ where A is an arbitrary scalar quantity is gauge invariant. Since \dot{A}/α where A is an arbitrary scalar quantity has the same Lie derivative as that of the scalar quantity A , $D^n(\dot{A}/\alpha)$ is also gauge invariant. In the way similar to the above calculation, as for \mathcal{A}_n defined by (2.73) we can prove

$$\frac{d}{d\mu} \mathcal{A}_n = L(T) \mathcal{A}_n. \tag{C.6}$$

Then \mathcal{A}_n is gauge invariant.

§D Proof of Proposition 2

Since ζ_1 is conserved for arbitrary values of the integration constants $C(\lambda = 0)$, ζ_1 with $C(\lambda = 0)$ replaced with $C(\lambda)$ is also conserved, that is, ζ_1 is conserved for arbitrary values of λ :

$$\left(\frac{\partial}{\partial a} \zeta_1 \right)_a = 0. \tag{D.1}$$

Then by the expression of $\bar{\delta}$ (2.80) we obtain

$$\zeta_2 = \left(\frac{\partial}{\partial \lambda} \zeta_1 \right)_a. \tag{D.2}$$

Since $(\partial/\partial a)_a$ and $(\partial/\partial \lambda)_a$ are commutative, we obtain

$$\left(\frac{\partial}{\partial a}\zeta_2\right)_a = \left(\frac{\partial}{\partial a}\frac{\partial}{\partial \lambda}\zeta_1\right)_a = \left(\frac{\partial}{\partial \lambda}\frac{\partial}{\partial a}\zeta_1\right)_a = 0. \quad (\text{D.3})$$

By iterating the same process, we obtain

$$\zeta_n = \left(\frac{\partial}{\partial \lambda}\right)_a^{n-1} \zeta_1, \quad \left(\frac{\partial}{\partial a}\zeta_n\right)_a = 0, \quad (\text{D.4})$$

for $n \geq 2$.

§E Proof of Proposition 3

When $P = P(\rho)$,

$$a \frac{\partial \rho}{\partial a} = -3(\rho + P). \quad (\text{E.1})$$

Then

$$\begin{aligned} \zeta_1 &= - \left\{ \left(\frac{\partial \rho}{\partial \lambda}\right)_a / \left(\frac{\partial \rho}{\partial a}\right)_a \right\} \frac{1}{a} \\ &= \frac{1}{3} \frac{1}{\rho + P} \left(\frac{\partial \rho}{\partial \lambda}\right)_a \\ &= \left(\frac{\partial}{\partial \lambda}\right)_a \left[\frac{1}{3} \int d\rho \frac{1}{\rho + P(\rho)} \right], \end{aligned} \quad (\text{E.2})$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial a}\zeta_1\right)_a &= \left(\frac{\partial}{\partial \lambda}\frac{\partial}{\partial a}\right)_a \left[\frac{1}{3} \int d\rho \frac{1}{\rho + P(\rho)} \right] \\ &= \left(\frac{\partial}{\partial \lambda}\right)_a \left(-\frac{1}{a}\right) \\ &= 0. \end{aligned} \quad (\text{E.3})$$

since $(\partial/\partial \lambda)_a$ and $(\partial/\partial a)_a$ are commutative. Then we can use Proposition 2, we complete the proof.

§F Proof of Proposition 4

For $k = 0$, by solving (3.11) (3.12) we obtain

$$|p - e^{-3N} p(0)| \leq \eta^2, \quad (\text{F.1})$$

$$|\phi - \phi(0)| \leq \eta^2 N. \quad (\text{F.2})$$

If $p(0), \phi(0) \sim 1$, Proposition 4 is right for $k = 0$. We assume that Proposition 4 holds for $k = 0, 1, 2, \dots, k-1$:

$$\left| \left(\frac{\partial^i}{\partial \lambda^i}\right)_a \phi \right|, \quad \left| \left(\frac{\partial^i}{\partial \lambda^i}\right)_a p \right| \sim \delta_c^i, \quad (i = 0, 1, 2, \dots, k-1), \quad (\text{F.3})$$

therefore

$$\left| \left(\frac{\partial^i}{\partial \lambda^i} \right)_a f(\phi, p, c_T, N) \right| \sim \delta_c^i, \quad (i = 0, 1, 2, \dots, k-1), \quad (\text{F.4})$$

if

$$|f(\phi, p, c_T, N)| \leq 1, \quad (\text{F.5})$$

for a proper complex domain containing the real interval which we consider. For k , the evolution equations are

$$\frac{d}{dN} \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi = \eta^2 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi + \eta^2 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p + \eta^2 \delta_c^k, \quad (\text{F.6})$$

$$\frac{d}{dN} \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p = -3 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p + \eta^2 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi + \eta^2 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p + \eta^2 \delta_c^k, \quad (\text{F.7})$$

where all the coefficients bounded by positive constants are omitted except -3 in (F.7). By solving (F.6)(F.7), for $0 \leq N \leq 1/\eta^2$, we obtain

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi \right| \leq \left| \frac{\partial^k}{\partial \lambda^k} \phi(0) \right| + \eta^2 \left| \frac{\partial^k}{\partial \lambda^k} p(0) \right| + \delta_c^k, \quad (\text{F.8})$$

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p \right| \leq e^{-3N} \left| \frac{\partial^k}{\partial \lambda^k} p(0) \right| + \eta^2 \left[\left| \frac{\partial^k}{\partial \lambda^k} \phi(0) \right| + \eta^2 \left| \frac{\partial^k}{\partial \lambda^k} p(0) \right| + \delta_c^k \right], \quad (\text{F.9})$$

Therefore for the initial conditions (3.17), for k , (3.18)(3.19) hold. By induction, for all nonnegative integers k , Proposition 4 holds.

§G Proof of Proposition 5

First we consider $k = 0$ case. By the mean value theorem, we obtain

$$\frac{d}{dN} \Delta \phi = \eta^2 \Delta \phi + \eta^2 f, \quad (\text{G.1})$$

where

$$|f| \leq |p| + \eta^2 \leq e^{-3N} + \eta^2, \quad (\text{G.2})$$

where we used Proposition 4 in the last inequality. By solving (G.1), we obtain

$$|\Delta \phi| \leq \exp(\eta^2 N) (|\Delta \phi(0)| + \eta^2 + \eta^4 N). \quad (\text{G.3})$$

Therefore for $0 \leq N \leq 1/\eta^2$, under the initial condition (3.22), (3.21) holds for $k = 0$. Next we consider a positive integer k case. In this case, we notice the following fact. By Proposition 4, if an complex analytic function f satisfies

$$|f(\phi, p, c_T, N)| \leq 1, \quad (\text{G.4})$$

then

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a f(\phi, p, c_T, N) \right| \leq \delta_c^k, \quad (\text{G.5})$$

By differentiating (G.1) with respect to λ and by using the above fact and Proposition 4, we obtain

$$\frac{d}{dN} \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \Delta \phi = \eta^2 \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \Delta \phi + \eta^2 \delta_c^k (e^{-3N} + \eta^2), \quad (\text{G.6})$$

which is solved as

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \Delta \phi \right| \leq \exp(\eta^2 N) \left(\left| \frac{\partial^k}{\partial \lambda^k} \Delta \phi(0) \right| + \eta^2 \delta_c^k + \eta^4 N \delta_c^k \right). \quad (\text{G.7})$$

Therefore for $0 \leq N \leq 1/\eta^2$, under the initial condition (3.22), (3.21) holds for a positive integer k .

§H Proof of Proposition 6

Lemma 1

$$\phi_a = \phi_a(0) \exp[-m_a^2 \tau] + \langle \mu_c \phi_0 \exp[-m^2 \tau] \rangle. \quad (\text{H.1})$$

Proof We consider the evolution equation

$$\frac{\partial}{\partial \tau} \phi_a = -m_a^2 \phi_a + \tilde{f}_a(\phi), \quad (\text{H.2})$$

where $\tilde{f}_a(\phi)$ is the sum of m -th order monomials ($m \geq 2$) satisfying

$$|\tilde{f}_a(\phi)| \leq m^2 \mu_c \frac{1}{\phi_0} |\phi|^2, \quad (\text{H.3})$$

for $|\phi| \leq \phi_0$. From (H.2), we obtain

$$\frac{\partial}{\partial \tau} |\phi| \leq -m^2 |\phi| + \frac{m^2 \mu_c}{\phi_0} |\phi|^2, \quad (\text{H.4})$$

which is solved as

$$|\phi| \leq \phi_0 \exp[-m^2 \tau] (1 + O(\mu_c)). \quad (\text{H.5})$$

We consider $\bar{\phi}_a$ satisfying

$$\frac{\partial}{\partial \tau} \bar{\phi}_a = -m_a^2 \bar{\phi}_a, \quad (\text{H.6})$$

$$\bar{\phi}_a(0) = \phi_a(0). \quad (\text{H.7})$$

The difference $\Delta \phi_a := \phi_a - \bar{\phi}_a$ satisfies

$$\frac{\partial}{\partial \tau} \Delta \phi_a = -m_a^2 \Delta \phi_a + \tilde{f}_a(\phi), \quad (\text{H.8})$$

$$\Delta \phi_a(0) = 0, \quad (\text{H.9})$$

where

$$|\tilde{f}_a(\phi)| \leq m^2 \mu_c \phi_0 \exp[-m^2 \tau], \quad (\text{H.10})$$

where we used (H.5). By solving the above evolution equation under the above initial condition, we obtain

$$|\Delta \phi| \leq O(\mu_c) \phi_0 \exp[-m^2 \tau]. \quad (\text{H.11})$$

We complete the proof. ■

Lemma 2

$$\left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \phi_a = \frac{\partial^k}{\partial \lambda^k} \phi_a(0) \cdot \exp[-m_a^2 \tau] + \langle \delta_c^k \mu_c \phi_0 \exp[-m^2 \tau] \rangle. \quad (\text{H.12})$$

Proof As for the evolution equation as

$$\frac{\partial}{\partial \tau} \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \phi_a = -m_a^2 \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \phi_a + \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \tilde{f}_a(\phi), \quad (\text{H.13})$$

where

$$\begin{aligned} \left| \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \tilde{f}_a(\phi) \right| &\leq m^2 \frac{\mu_c}{\phi_0} |\phi| \left| \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \phi_a \right| + m^2 \frac{\mu_c}{\phi_0} \delta_c^k |\phi|^2 \\ &\leq m^2 \mu_c \exp[-m^2 \tau] \left| \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \phi_a \right| + \delta_c^k \mu_c m^2 \phi_0 \exp[-2m^2 \tau]. \end{aligned} \quad (\text{H.14})$$

We perform the calculations similar to the proof of Lemma 1. We complete the proof. ■

Lemma 3

$$\left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \left(\prod_{l=1}^N \phi_{a(l)} \right) = \frac{\partial^k}{\partial \lambda^k} \left(\prod_{l=1}^N \phi_{a(l)}(0) \right) \exp \left[- \sum_{l=1}^N m_{a(l)}^2 \tau \right] + \langle \delta_c^k \mu_c \phi_0^N \exp[-Nm^2 \tau] \rangle. \quad (\text{H.15})$$

Proof We use the Leibniz rule for the λ derivative and use Lemma 1, Lemma 2. We complete the proof. ■

Lemma 4 For $n \geq 1$

$$A(2n, 0) = \sum_a (m_a^2)^n \phi_a^2 + F_{2n}(\phi) \quad (\text{H.16})$$

where $F_{2n}(\phi)$ is the sum of m -th order monomials of ϕ ($m \geq 3$) and satisfies

$$|F_{2n}(\phi)| \leq \mu_c m^{2n} \frac{1}{\phi_0} |\phi|^3 \quad (\text{H.17})$$

for $|\phi| \leq \phi_0$.

Proof We use (H.2). We complete the proof. ■

As for

$$\left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau \bar{N} = \kappa^2 \int_0^\infty d\tau \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau U, \quad (\text{H.18})$$

and

$$A(0, k) = 4 \int_\tau^\infty d\tau \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau U, \quad (\text{H.19})$$

and

$$A(2n, k) = \left(\frac{\partial^k}{\partial \lambda^k}\right)_\tau A(2n, 0), \quad (\text{H.20})$$

for $n \geq 1$ where $A(2n, 0)$ is expressed as a polynomial of $\phi(\tau)$ (H.16), we use Lemma 3. Then we complete the proof of Proposition 6.

§I Proof of Proposition 7

By α_M, α_m , we mean

$$\alpha_M := \max_a \{m_a^2\}, \quad (\text{I.1})$$

$$\alpha_m := \min_a \{m_a^2\}. \quad (\text{I.2})$$

Then we obtain

$$|(k \cdot \alpha) - \alpha_a| \geq |k| \alpha_m - \alpha_M. \quad (\text{I.3})$$

Therefore for k satisfying

$$|k| \geq \frac{\delta_1 + \alpha_M}{\alpha_m} =: L, \quad (\text{I.4})$$

where δ_1 is a positive constant, we obtain

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_1. \quad (\text{I.5})$$

Since the number of k satisfying $|k| < L$ is finite, there exists a positive constant δ_2 satisfying

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_2, \quad (\text{I.6})$$

for $|k| < L$. Therefore for δ_m defined by

$$\delta_m := \min\{\delta_1, \delta_2\}, \quad (\text{I.7})$$

the inequality (3.71) holds. We complete the proof.

§J Proof of Proposition 8

By replacing ϕ_a with $\phi_0 \phi_a$ and by replacing τ with $\tau/\mu_c \alpha_M$, we can assume that $|\tilde{f}_a(\phi)| \leq 1$ for $|\phi| \leq 1$. In order that the transformation law (3.74) can reduce the original equation (3.72) to the exactly linear equation (3.76), w_a must satisfy

$$-\frac{\partial w_a}{\partial \varphi_b} \alpha_b \varphi_b + \alpha_a w_a = \tilde{f}_a(\varphi + w). \quad (\text{J.1})$$

We decompose w_a into the sum of the K -th order polynomials:

$$w_a = \sum_{K \geq 2} w_a^K, \quad w_a^K = \sum_{|k|=K} w_a^k \varphi^k, \quad (\text{J.2})$$

where

$$\varphi^k := \varphi_1^{k_1} \cdots \varphi_{N_S}^{k_{N_S}}, \quad |k| := k_1 + \cdots + k_{N_S}. \quad (\text{J.3})$$

We decompose (J.1) into K -th order part as

$$-\frac{\partial w_a^K}{\partial \varphi_b} \alpha_b \varphi_b + \alpha_a w_a^K = P_a^K(\tilde{f}, w^1, \dots, w^{K-1}), \quad P_a^K = \sum_{|k|=K} d_a^k \varphi^k \quad (\text{J.4})$$

for $K = 2, \dots$. We mean the coefficient of φ^k in \tilde{f}_a by f_a^k . Then d_a^k ($|k| = K$) is the polynomial of f_a^k ($|k| \leq K$) and w_a^k ($|k| \leq K - 1$) with integral coefficients and w_a^k can be expressed as

$$w_a^k = \frac{d_a^k}{-(k \cdot \alpha) + \alpha_a}. \quad (\text{J.5})$$

From now on, we will prove the convergence of the transformation w_a by the method of the majorant. $\tilde{F}(\varphi)$ defined by

$$\tilde{F}(\varphi) = \sum_{K=2}^{\infty} (\varphi_1 + \dots + \varphi_{N_S})^K \quad (\text{J.6})$$

is the majorant of \tilde{f}_a . We consider the equation as

$$\delta_m \cdot \tilde{W}(\varphi) = \tilde{F}(\varphi + \tilde{W}(\varphi)), \quad (\text{J.7})$$

where δ_m is defined in Proposition 7, Since $\tilde{F}(\varphi)$ is a independent, $\tilde{W}_a(\varphi)$ is a independent. Then $\tilde{W}_a(\varphi)$ is simply written as $\tilde{W}(\varphi)$. We will prove that $\tilde{W}(\varphi)$ is the majorant of $w_a(\varphi)$. By expanding $\tilde{W}(\varphi)$, $\tilde{F}(\varphi)$ $\tilde{F}(\varphi + \tilde{W}(\varphi))$ as

$$\tilde{W}(\varphi) = \sum_{|k| \geq 2} W^k \varphi^k, \quad \tilde{F}(\varphi) = \sum_{|k| \geq 2} F^k \varphi^k, \quad \tilde{F}(\varphi + \tilde{W}(\varphi)) = \sum_{|k| \geq 2} D^k \varphi^k, \quad (\text{J.8})$$

we obtain $W^k = D^k / \delta_m$. For $|k| = 2$, $D^k = F^k$, so $|w_a^k| \leq W^k$. We assume that $|w_a^k| \leq W^k$ for $|k| \leq K - 1$. Since d_a^k ($|k| = K$) is the polynomial of f_a^k ($|k| \leq K$) and w_a^k ($|k| \leq K - 1$) with integral coefficients and D^k is given by substitution $f_a^k \rightarrow F^k$, $w_a^k \rightarrow W^k$ in the polynomial representing d_a^k , $|d_a^k| \leq D^k$ ($|k| = K$). Therefore we obtain $|w_a^k| \leq W^k$ ($|k| = K$). By induction, $\tilde{W}(\varphi)$ is the majorant of $w_a(\varphi)$. As for the right hand side of (J.7),

$$\tilde{F}(\varphi + \tilde{W}(\varphi)) = \frac{(\Phi + N_S \tilde{W}(\varphi))^2}{1 - (\Phi + N_S \tilde{W}(\varphi))}, \quad \Phi := \varphi_1 + \dots + \varphi_{N_S}, \quad (\text{J.9})$$

so long as $|\Phi + N_S \tilde{W}(\varphi)| < 1$. So (J.7) can be written as

$$N_S(N_S + \delta_m) \tilde{W}^2 - [\delta_m - (2N_S + \delta_m)\Phi] \tilde{W} + \Phi^2 = 0, \quad (\text{J.10})$$

whose solution is given by

$$\tilde{W} = \frac{1}{2N_S(N_S + \delta_m)} \left[\delta_m - (2N_S + \delta_m)\Phi - \sqrt{\delta_m^2 - 2\delta_m(2N_S + \delta_m)\Phi + \delta_m^2 \Phi^2} \right] \quad (\text{J.11})$$

for $|\Phi + N_S \tilde{W}(\varphi)| < 1$, using the fact that $\tilde{W} = 0$ for $\varphi_a = 0$. Since \tilde{W} is the majorant of w_a , we can prove the convergence of w_a .

This proof in Appendix J is based on [21].

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